Specht Modules and Schubert Varieties for General Diagrams

by

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Abstract

The algebra of symmetric functions, the representation theory of the symmetric group, and the geometry of the Grassmannian are related to each other via Schur functions, Specht modules, and Schubert varieties, all of which are indexed by partitions and their Young diagrams. We will generalize these objects to allow for not just Young diagrams but arbitrary collections of boxes or, equally well, bipartite graphs. We will then provide evidence for a conjecture that the relation between the areas described above can be extended to these general diagrams.

In particular, we will prove the conjecture for forests. Along the way, we will use a novel geometric approach to show that the dimension of the Specht module of a forest is the same as the normalized volume of its matching polytope. We will also demonstrate a new Littlewood-Richardson rule and provide combinatorial, algebraic, and geometric interpretations of it.

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Chapter 1

Introduction

One of the most surprising yet fundamental results in algebraic combinatorics is the connection between the algebra of symmetric functions, the representation theory of the symmetric group, and the geometry of the Grassmannian. At the heart of this connection is the role of the partition and its Young diagram, which ties together the main objects in all three of these areas. The main goal of this thesis is to show that one can formulate a similar connection for certain diagrams other than partitions and to provide evidence for an even more general relationship that has yet to be seen.

In this chapter, we will first review the basic connection between the three areas described above. We will then describe how to generalize the main objects of study to allow for diagrams other than partitions. Finally, we will present the main conjecture that will provide the motivation for the rest of this thesis.

1.1 Preliminaries

In this section, we review the basic preliminaries needed for our work. (For further references, see, for instance, [6], [22], [23].)

1.1.1 Symmetric functions

Consider an array of unit lattice boxes in the plane. We will write (i, j) to denote the box in the ith row from the top and the jth column from the left, where i and j are positive integers. By a *diagram*, we will mean any finite subset of these boxes.

For instance, a *partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ of $|\lambda| = n$ is a sequence of weakly decreasing nonnegative integers summing to n. (We may add or ignore trailing zeroes to λ as convenient.) Then the *Young diagram* of λ consists of all boxes (i, j) with $j \leq \lambda_i$. We may refer to a partition and its Young diagram interchangeably. If λ and μ are partitions such that $\mu_i \leq \lambda_i$ for all i, then we write $\mu \subset \lambda$, and the *skew Young diagram* λ/μ consists of all boxes in λ not in μ.

For any diagram D, a *tableau* T of shape D is a filling of the boxes of D with nonnegative integers. In the case of a Young diagram λ (or a skew Young diagram λ/μ), we say T is a *semistandard Young tableau (SSYT)* if its rows are weakly increasing and its columns are strictly increasing. The *weight* (or *type*) of a tableau T is the sequence $\alpha = (\alpha_1, \alpha_2, \dots)$, where α_i is the number of occurrences of i in T. (Again, we may add or ignore trailing zeroes to α as convenient.) A semistandard tableau T of shape λ (or λ/μ) is a *standard Young tableau (SYT)* if its weight is $(1, 1, \dots, 1)$.

For any commutative ring R, let Λ_R denote the ring of symmetric functions over commuting variables $x_1, x_2, \dots$ with coefficients in R. We will usually take $R = \mathbf{C}$, and we will write Λ for the $\mathbf{C}$-vector space $\Lambda_{\mathbf{C}}$.

To each Young diagram λ, we associate a homogeneous symmetric function $s_\lambda \in \Lambda_{\mathbf{Z}}$ of degree $|\lambda|$, called the *Schur function* associated to λ, as follows:

$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\alpha_1} x_2^{\alpha_2} \cdots ,$$

where the sum ranges over all semistandard tableaux T of shape λ, and α is the weight of T. Note that the coefficient of the squarefree term $x_1 x_2 \cdots x_n$ in s_λ (where $n = |\lambda|$) is f^λ, the number of standard Young tableaux of shape λ.

It is not immediately clear from this definition that s_λ is a symmetric function. However, one can even say the following.

Proposition 1.1.1. *The Schur functions $\{s_\lambda\}$, as λ ranges over all partitions, form a linear basis for the ring of symmetric functions Λ and an integer basis for $\Lambda_{\mathbf{Z}}$.*

The *Littlewood-Richardson coefficients* $c^\lambda_{\mu\nu}$ are defined to be the structure constants of Λ with respect to the basis $\{s_\lambda\}$. In other words:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

Clearly $c^\lambda_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$. In fact, these coefficients are always nonnegative integers. There are a number of ways to determine $c^\lambda_{\mu\nu}$ combinatorially, and these are generally referred to collectively as *Littlewood-Richardson rules* (see, for instance, [12], [24]). Moreover, these coefficients have a number of symmetries, which may or may not be obvious from a given rule.

For instance, fix positive integers $k < r$, and let $k \times (r-k)$ be the rectangular Young diagram with k parts of size $r-k$. For $\lambda \subset k \times (r-k)$, let

$$\lambda^\vee = (r-k-\lambda_k, r-k-\lambda_{k-1}, \ldots, r-k-\lambda_1),$$

so that $\lambda^\vee$ and λ can be fit together to form the entire $k \times (r-k)$ rectangle. Then the Littlewood-Richardson coefficients have the following symmetry.

Proposition 1.1.2. *The Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$ is symmetric in μ, ν, and $\lambda^\vee$.*

For instance, it follows that $c^\lambda_{\mu\nu} = c^{\mu^\vee}_{\lambda^\vee\nu}$.

Perhaps surprisingly, the Littlewood-Richardson coefficients also appear in a similar but distinct context. Given a skew Young diagram λ/μ, we can define the *skew Schur function* $s_{\lambda/\mu}$ as above:

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\alpha_1} x_2^{\alpha_2} \cdots ,$$

where the sum ranges over all semistandard tableaux T of shape λ/μ, and α is the weight of T. If we then ask how $s_{\lambda/\mu}$ may be written in terms of the Schur basis, the answer again involves Littlewood-Richardson coefficients.

Proposition 1.1.3.

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

Next, we will describe another occurrence of the Littlewood-Richardson coefficients, this time in the representation theory of the symmetric group.

1.1.2 Representation theory of the symmetric group

Let Σ_n denote the symmetric group on n letters. Like all finite groups, the number of irreducible representations of Σ_n is equal to the number of conjugacy classes of Σ_n, which are given by partitions of n. In this case, one can canonically index the irreducible representations of Σ_n by partitions. There are a variety of constructions for these representations (see, for instance, [1]), but below we describe one of the first constructions, as well as the one that will generalize most naturally to other diagrams.

Consider any Young diagram λ, $|\lambda| = n$. Order the boxes of λ arbitrarily, and let Σ_n act on them in the usual way. Let R_λ be the subgroup containing those $\sigma \in \Sigma_n$ that stabilize the rows of λ, and likewise define C_λ for columns of λ. Let $\mathbf{C}[\Sigma_n]$ denote the group algebra over Σ_n with complex coefficients, and consider elements

$$R(\lambda) = \sum_{\sigma \in R_\lambda} \sigma \text{ and } C(\lambda) = \sum_{\sigma \in C_\lambda} \operatorname{sgn}(\sigma)\sigma.$$

The *Specht module* (over $\mathbf{C}$) of λ is the left ideal

$$S^\lambda = \mathbf{C}[\Sigma_n]C(\lambda)R(\lambda).$$

Clearly S^λ is a representation of Σ_n by left multiplication. We may interpret the elements of Σ_n in $\mathbf{C}[\Sigma_n]$ as bijections from the boxes of λ to $[n]$, or, equivalently, as tableaux of shape λ with labels $1, 2, \ldots, n$. Then multiplying on the right corresponds to applying a permutation to the boxes of λ, while the action of Σ_n on the left corresponds to applying a permutation to the labels $[n]$.

Proposition 1.1.4. *As λ ranges over all partitions of n, the Specht modules S^λ are exactly the irreducible representations of Σ_n. Moreover, as T ranges over standard Young tableaux of shape λ, the elements $T \cdot C(\lambda)R(\lambda)$ form a linear basis of S^λ. In particular,* $\dim S^\lambda = f^\lambda$*, the number of standard Young tableaux of shape λ.*

Suppose μ is a partition of m and ν is a partition of n. Then one can form the outer tensor product representation $S^\mu \otimes S^\nu$ of $\Sigma_m \times \Sigma_n$. Embedding $\Sigma_m \times \Sigma_n$ in Σ_{m+n} in the usual way, one can ask how the induced representation, which we denote $S^\mu \circ S^\nu$, decomposes into irreducible representations.

Proposition 1.1.5. *Let μ and ν be partitions of m and n, respectively. Then*

$$S^\mu \circ S^\nu = \operatorname{Ind}_{\Sigma_m \times \Sigma_n}^{\Sigma_{m+n}} S^\mu \otimes S^\nu \cong \bigoplus_\lambda (S^\lambda)^{\oplus c^\lambda_{\mu\nu}},$$

where $c^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients.

Another way of describing the appearance of the Littlewood-Richardson coefficients is to use the *(Frobenius) characteristic map.* Let R^n denote the lattice of virtual characters (differences of characters) of Σ_n. Then $R = R^0 \oplus R^1 \oplus R^2 \oplus \cdots$ can be given a ring structure as follows: if $\chi^V \in R^m$ and $\chi^W \in R^n$ are characters of representations V and W, respectively, then $\chi^V \circ \chi^W \in R^{m+n}$ is the character of $V \circ W$. We then define $\mathrm{ch}\colon R \to \Lambda_{\mathbf{Z}}$ via $\mathrm{ch}(\chi^\lambda) = s_\lambda$, where χ^λ is the character of S^λ. (We may sometimes abusively write $\mathrm{ch}(V) = \mathrm{ch}(\chi^V)$.) Then the previous proposition essentially gives the following.

Proposition 1.1.6. *The characteristic map* $\mathrm{ch}\colon R \to \Lambda_{\mathbf{Z}}$ *is a ring isomorphism.*

Just as one can define skew Schur functions for skew Young diagrams, one can also define Specht modules for skew Young diagrams: given a skew Young diagram λ/μ with $|\lambda| - |\mu| = n$, define $R_{\lambda/\mu}$, $C_{\lambda/\mu}$, $R(\lambda/\mu)$, and $C(\lambda/\mu)$ in the analogous way as for ordinary Young diagrams, and let

$$S^{\lambda/\mu} = \mathbf{C}[\Sigma_n]C(\lambda/\mu)R(\lambda/\mu).$$

We then have the following analogous result for skew Young diagrams.

Proposition 1.1.7. *As T ranges over standard Young tableaux of shape λ/μ, the elements $T \cdot C(\lambda/\mu) R(\lambda/\mu)$ form a linear basis of $S^{\lambda/\mu}$. In particular, $\dim S^{\lambda/\mu} = f^{\lambda/\mu}$, the number of standard Young tableaux of shape λ. Moreover, $\mathrm{ch}(S^{\lambda/\mu}) = s_{\lambda/\mu}$, so that*

$$S^{\lambda/\mu} \cong \bigoplus_{\nu} (S^{\nu})^{\oplus c^{\lambda}_{\mu\nu}}.$$

Next, we describe one final occurrence of the Littlewood-Richardson coefficients, this time with regards to the geometry of the Grassmannian.

1.1.3 Geometry of the Grassmannian

Let $G(k,r)$ denote the Grassmannian of k-dimensional (complex) subspaces in an r-dimensional complex vector space V. Fix a flag $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = V$, with $\dim V_i = i$. For $\lambda \subset k \times (r-k)$, we associate the *Schubert cell*

$$\Omega^{\circ}_{\lambda} = \{L \in G(k,r) \mid \dim(L \cap V_j) = i \text{ for } r-k+i-\lambda_i \leq j \leq r-k+i-\lambda_{i+1}, 0 \leq i \leq k\}.$$

In coordinates, we can express an element of $G(k,r)$ via a $k \times r$ matrix of full rank up to left action by $GL(k, \mathbf{C})$. Let us choose a basis $e_1, e_2, \ldots, e_r$ of V such that V_i is the span of $e_1, \ldots, e_i$. The condition for $L \in \Omega^{\circ}_{\lambda}$ means that the row-reduction of the matrix for L from right to left has pivots in columns $r-k+i-\lambda_i$ for $1 \leq i \leq k$. In particular, by changing the order of the basis $\{e_i\}$, we can write the matrix of L in the form $(I_k \mid B)$, where I_k is the identity matrix, and B is a $k \times (r-k)$ matrix with zeroes in the entries corresponding to boxes of λ.

Let the *Schubert variety* Ω_{λ} be the Zariski closure of Ω°_{λ}. From the discussion above, it is easy to see that $\dim \Omega_{\lambda} = k(r-k) - |\lambda|$, so Ω_{λ} defines a *Schubert class* $\sigma_{\lambda} \in H^{2|\lambda|}(G(k,r))$ (we take cohomology with integer coefficients). Then we have the following description of $H^*(G(k,r))$.

Proposition 1.1.8. *The cohomology ring $H^*(G(k,r))$ is isomorphic to $\Lambda_{\mathbf{Z}}/I$, where I is the ideal generated by all s_{λ} such that $\lambda \not\subset k \times (r-k)$. Moreover, the isomorphism* *is given by* $\sigma_\lambda \mapsto s_\lambda$*. In particular,* $\{\sigma_\lambda \mid \lambda \subset k \times (r-k)\}$ *forms an additive* $\mathbf{Z}$*-basis for* $H^*(G(k,r))$*, and*

$$\sigma_\mu \smile \sigma_\nu = \sum_{\lambda \subset k \times (r-k)} c^\lambda_{\mu\nu} \sigma_\lambda,$$

where the $c^\lambda_{\mu\nu}$ *are the Littlewood-Richardson coefficients.*

It is also known that the class of any subvariety can be written as a nonnegative linear combination of the σ_λ.

This proposition facilitates the basic calculations in Schubert calculus. For instance, under the Plücker embedding, the hyperplane class in $H^2(G(k,r))$ is given by σ_1. Therefore, by multiplying by the appropriate power of σ_1, one can calculate the degree of a Schubert variety, giving the following proposition. (Whenever we consider degrees of subvarieties of $G(k,r)$, we consider them under the Plücker embedding.)

Proposition 1.1.9.

$$\deg \Omega_{\lambda^\vee} = f^\lambda.$$

Given the relationship between the three areas described above, it makes sense to ask how one may be able to generalize it. We will describe the primary generalization considered in this thesis in the next section.

1.2 General diagrams

In the discussion above, we described how the Specht module S^λ and the Schubert variety Ω_λ play analogous roles via their connection to the Schur function s_λ. We now wish to generalize the role of λ to arbitrary diagrams.

Recall that a diagram D is an arbitrary finite collection of unit lattice boxes in the plane. We begin by defining the Specht module S^D. Notice that the definition given for Specht modules S^λ (and $S^{\lambda/\mu}$) does not use the fact that λ is a (skew) Young diagram. As such, we can apply exactly the same definition to any diagram D.

Explicitly, let $n = |D|$. Order the boxes of D arbitrarily, and let Σ_n act on them in the usual way. As before, let R_D and C_D be the subgroups containing those $\sigma \in \Sigma_n$

that stabilize each row of D and each column of D, respectively, and let

$$R(D) = \sum_{\sigma \in R_D} \sigma \text{ and } C(D) = \sum_{\sigma \in C_D} \operatorname{sgn}(\sigma)\sigma.$$

Then the *Specht module* of D is the left ideal

$$S^D = \mathbf{C}[\Sigma_n] C(D) R(D).$$

We will also write χ^D for the character of S^D.

Such Specht modules have been well studied. For instance, it is not hard to show that if $D \subset k \times (r-k)$, then $c_\lambda^D = 0$ unless $\lambda \subset k \times (r-k)$. However, they are still not completely understood except in special cases. (See, for example, [21].)

Magyar [17] showed that Specht modules behave nicely under box complementation in the following sense. For any diagram $D \subset k \times (r-k)$ rectangle, let $D^\vee$ be the complement of D in this rectangle. (Although this is not quite consistent with the notation $\lambda^\vee$ above, the extra half-turn will usually be unimportant.)

Proposition 1.2.1 (Magyar [17])**.**

$$c_\lambda^D = c_{\lambda^\vee}^{D^\vee}.$$

Next, we define the *Schubert variety* Ω_D for a general diagram $D \subset k \times (r-k)$. Consider those $k \times r$ matrices of the form $A = (I_k \mid B)$, where I_k is the identity matrix and B is a $k \times (r-k)$ matrix with zero in the entries corresponding to boxes of D. Then these matrices define a quasiprojective subvariety of $G(k, r)$, which we denote Ω_D°. We then let Ω_D be the closure of Ω_D°. Finally, we let σ_D be the cohomology class of Ω_D. Note that this definition agrees with the usual definition when D is a partition λ.

One key property of these definitions is that they do not depend on the order of the rows and columns of D. In other words, applying a permutation to the rows of D or to the columns of D does not affect the structure of S^D up to isomorphism or the cohomology class σ_D. (In the first case, permuting does not change the row or column stabilizers, while in the second case, permuting corresponds simply to changing the order of basis elements.)

To put it another way, let us define a bipartite graph $G = G(D)$ as follows. The rows of D correspond to white vertices of G, the columns of D to black vertices of G, and the boxes of D to edges of G connecting the corresponding row and column. Then the structure of S^D and the class σ_D depend only on the (2-colored) bipartite graph. As such, it will sometimes be convenient to refer to the Specht module or Schubert class of a (2-colored) bipartite graph G, which we will denote by S^G and σ_G. We may also abusively refer to G and D interchangeably (such as by referring to the diagram G, or by saying that D is a forest). We will say that two diagrams are equivalent if their corresponding (2-colored) bipartite graphs are isomorphic. We may also at times implicitly identify equivalent diagrams if the order of rows and columns is unimportant.

1.3 Main conjecture

We now formulate the main conjecture that will guide this thesis. In short, it states that for any diagram D, χ^D and σ_D are essentially the same.

Explicitly, let c_λ^D be the multiplicity of S^λ in S^D, and define the *Schur function* associated to D by

$$s_D = \mathrm{ch}(\chi^D) = \sum_\lambda c_\lambda^D s_\lambda.$$

Conjecture 1. *For all diagrams D,*

$$\sigma_D = \sum_\lambda c_\lambda^D \sigma_\lambda.$$

In other words, let $\varphi\colon \Lambda_{\mathbf{Z}} \to H^*(G(k,r))$ be the map sending $s_\lambda \mapsto \sigma_\lambda$ if $\lambda \subset k \times (r-k)$ and $s_\lambda \mapsto 0$ otherwise. Then if D fits inside a $k \times (r-k)$ rectangle, the conjecture states that $\varphi(s_D) = \sigma_D$. As a remark, we have not explicitly chosen k and r with which to define σ_D, but it is not hard to check that the expression for σ_D in terms of σ_λ does not depend on this choice. (We will show this in Chapter 5.)

While we will not be able to present a proof of this conjecture here, we will nonetheless provide several pieces of evidence for it, including the special case when $D^\vee$ is a forest.

By combining Propositions 1.2.1 and 1.1.9 above (and replacing D by $D^\vee$), we arrive at the following slightly weaker conjecture:

Conjecture 2. *For all diagrams D,* $\dim S^D = \deg \Omega_{D^\vee}$.

By direct computation, we have verified Conjecture 2 for all diagrams with at most eight boxes. (There are 305 connected such diagrams. Appendix A lists $\dim S^D$ and s_D for those D that are not skew Young diagrams.)

One of our main results will be to provide a combinatorial interpretation for this quantity when D is a forest, which will include an unexpected detour into the study of matching polytopes.

The next chapter will be devoted to the structure of general Specht modules. In Chapter 3, we define the notion of the matching polytope of a bipartite graph and present recurrences for calculating its volume. These two chapters come together in Chapter 4, where we show that the dimension of the Specht module for a forest is equal to the normalized volume of its matching polytope. In Chapter 5, we discuss the structure of general Schubert varieties, and we prove the special case of our main conjecture when $D^\vee$ is a forest, as well as some other special cases of Conjecture 2. In Chapter 6, we present a new Littlewood-Richardson rule that highlights the relationship between the geometric and algebraic objects of study. Finally, we end with a discussion of possible directions for future work.

Chapter 2

Specht and Schur Modules

In this chapter, we review some of the known facts about Specht modules and define the related representations of $GL(N)$ called Schur modules. (When we refer to representations of $GL(N)$, we restrict our attention to polynomial representations over $\mathbf{C}$.) We then present two results on the structure of Specht modules.

First, we introduce the following notation. If V and W are representations of Σ_n (or $GL(N)$), we write $V \leq W$ if W contains a subrepresentation isomorphic to V. In particular, since for both Σ_n and $GL(N)$ we have complete reducibility of representations, $V \leq W$ if there exists an injection (of representations) $V \to W$ or a surjection $W \to V$.

2.1 Background

For convenience, we reproduce the definition of Specht modules here.

Definition. Let Σ_n be the symmetric group on the n boxes of a diagram D. Let R_D and C_D be the subgroups of Σ_n that stabilize the rows of D and the columns of D, respectively, and let

$$R(D) = \sum_{\sigma \in R_D} \sigma \text{ and } C(D) = \sum_{\sigma \in C_D} \operatorname{sgn}(\sigma)\sigma.$$

Then the *Specht module* S^D is the left ideal

$$S^D = \mathbf{C}[\Sigma_n]C(D)R(D).$$

Specht modules have been studied in a number of contexts (see, for instance, [14], [17], [18], [20], [21]). The most general result regarding their structure is due to Reiner and Shimozono [21].

Definition. We say that a diagram D is *percentage-avoiding* (or *%-avoiding*) if there do not exist boxes $(i_1, j_1), (i_2, j_2) \in D$ with $i_1 < i_2$ and $j_1 > j_2$ such that $(i_1, j_2), (i_2, j_1) \notin D$.

Reiner and Shimozono describe the structure of S^D when D is percentage-avoiding in terms of so-called *D-peelable tableaux*. The exact definition of D-peelable tableaux will not be important for our purposes.

The representation theory of the symmetric group is closely related to that of the general linear group. We therefore define the analogue of Specht modules, called Schur modules.

Definition. Let D be a diagram with n boxes and at most N rows. The *Schur module* $\mathscr{S}^D$ is a $GL(N)(= GL(N, \mathbf{C}))$-module defined as follows. Let $V = \mathbf{C}^N$ be the defining representation of $GL(N)$. Then

$$\mathscr{S}^D = V^{\otimes n}C(D)R(D),$$

where we associate the copies of V in the tensor power to the boxes of the diagram D, and $C(D)$ and $R(D)$ act in the obvious way.

Note that even if D has more than N rows, we can still make this definition, except that we may find $\mathscr{S}^D = 0$.

Let $e_1, \ldots, e_N$ be the standard basis of V. Then we may identify a tableau of shape D with entries at most N with the corresponding tensor product of standard basis elements of V, an element of $V^{\otimes n}$. Then $\mathscr{S}^D$ is spanned by $TC(D)R(D)$, where T ranges over all tableaux of shape D with entries at most N.

When D is the Young diagram of a partition λ with at most N parts, $\mathscr{S}^\lambda$ is the irreducible polynomial representation of $GL(N)$ with highest weight λ. In general, $\mathscr{S}^D$ is a polynomial representation of $GL(N)$ of degree n, and it follows from Schur-Weyl duality (see [8]) that

$$\mathscr{S}^D \cong \bigoplus (\mathscr{S}^\lambda)^{\oplus c_\lambda^D},$$

where the coefficients c_λ^D are the same as those encountered in the decomposition of the Specht module S^D.

Recall that the *character* of $\mathscr{S}^D$ is the trace of the action of the diagonal matrix $\operatorname{diag}(x_1, \ldots, x_N)$ on $\mathscr{S}^D$ as a polynomial in the x_i. Then it follows from the fact that $\operatorname{ch}(\mathscr{S}^\lambda) = s_\lambda$ and the definition of the Schur function s_D that

$$\operatorname{ch}(\mathscr{S}^D) = s_D(x_1, \ldots, x_N).$$

In particular, the dimension of $\mathscr{S}^D$ is the trace of the image of the identity, so we find that

$$\dim \mathscr{S}^D = s_D(\underbrace{1, 1, \ldots, 1}_{N}).$$

2.2 Splitting Specht modules

In this section, we present a method for (almost) splitting a Specht module into two pieces. Specifically, given a diagram D, we show how to construct diagrams D^A and D^B such that $S^{D^A} \oplus S^{D^B} \leq S^D$. Here we sometimes implicitly identify a row of D with the subset of columns containing a box in that row, so, for instance, in a Young diagram, the rows are ordered by reverse inclusion.

The results of this section are a slight generalization of those used by James and Peel [18]. First, an easy lemma.

Lemma 2.2.1. *Let E_1 and E_2 be two diagrams with n boxes, and let $\sigma\colon E_1 \to E_2$ be a bijection such that there exist two boxes x and y in the same row of E_1 with $\sigma(x) = z$ and $\sigma(y) = w$ lying in the same column of E_2. Then $C(E_2)\sigma R(E_1) = 0$.*

Proof. Note $\sigma \cdot (x, y) = (z, w) \cdot \sigma$, so that $(id - (z, w))\sigma(id + (x, y)) = 0$. Since we can factor $C(E_2) = C' \cdot (id - (z, w))$ and $R(E_1) = (id + (x, y)) \cdot R'$, it follows that

$$C(E_2)\sigma R(E_1) = C' \cdot (id - (z, w))\sigma(id + (x, y)) \cdot R' = 0. \qquad \square$$

We next describe a simple condition to tell whether a diagram is equivalent to a Young diagram.

Proposition 2.2.2. *A diagram D is equivalent to the diagram of a partition if and only if for every $(i_1, j_1), (i_2, j_2) \in D$, either (i_1, j_2) or (i_2, j_1) lies in D.*

Proof. The "only if" direction is easy. For the "if" direction, suppose D satisfies the second condition, and consider any two rows i_1 and i_2 that are not identical. Then without loss of generality, row i_1 contains a box of D in column j_1 but row i_2 does not. Then for any j_2, if $(i_2, j_2) \in D$, then we must have that $(i_1, j_2) \in D$, so row i_2 is contained in row i_1. It follows easily that the rows of D and likewise the columns of D can be ordered by reverse inclusion, yielding the diagram of a partition. $\square$

Suppose $(i_1, j_1), (i_2, j_2) \in D$ but $(i_1, j_2), (i_2, j_1) \notin D$. Let D^A be the diagram obtained from D by replacing row i_1 with the intersection of rows i_1 and i_2 and by replacing row i_2 with the union of rows i_1 and i_2. In other words, we move boxes in row i_1 to row i_2 within the same column if possible. We will say that D^A is formed by "collapsing" rows i_1 and i_2 of D.

Note that this gives an implicit bijection between the boxes of D and D^A. We use this bijection to define the action of Σ_n on D and D^A consistently.

Proposition 2.2.3. *There exists a surjective $\mathbf{C}[\Sigma_n]$-module homomorphism of Specht modules $T\colon S^D \to S^{D^A}$.*

Proof. Choose a set X of left coset representatives of $R_D \cap R_{D^A}$ in R_D and a set Y of right coset representatives of $R_D \cap R_{D^A}$ in R_{D^A}. Let T be the map from S^D to $\mathbf{C}[\Sigma_n]$ given by right multiplication by $\sum_{\sigma \in Y} \sigma$.

Suppose $\sigma \in R_D \backslash R_{D^A}$. (This holds for all but one element of X.) Then σ satisfies the condition of Lemma 2.2.1 for $E_1 = E_2 = D^A$, so $C(D^A)\sigma R(D^A) = 0$. Noting also that $C(D) = C(D^A)$, it follows that

$$C(D^A)R(D^A) = C(D)\left(\sum_{\sigma\in X}\sigma\right)R(D^A) = C(D)R(D)\sum_{\sigma\in Y}\sigma.$$

Thus the image of T lies in S^{D^A}, as desired. $\square$

In general it seems difficult to precisely determine the kernel of the map T except in special cases. However, there is something that can be said.

Let D^A be defined from D, i_1, i_2, j_1, and j_2 as above. Suppose that D^B is a diagram obtained in the following way. First, D^B differs from D only by moving each box $(a, j_1) \in D$ to some empty space $(a, f(a))$ (or possibly $f(a) = j_1$). Second, $f(i_1) = j_2$. Third, the rows a such that $(a, j_1) \in D$ can be ordered $a_1, a_2, \dots$ in such a way that for all $p < q$, $(a_q, f(a_p)) \in D^B$.

Although these conditions may seem a bit strange, we will usually take D^B to be the diagram obtained from D by moving the boxes in column j_1 to column j_2 within the same row if possible, that is, by "collapsing" columns j_1 and j_2. (It is easy to check that this satisfies the conditions above.) In general, D^B is obtained from B by collapsing first columns j_1 and $f(a_1)$, then columns j_1 and $f(a_2)$, and so forth.

Lemma 2.2.4. *Let D, D^A, and D^B be given as above. Then S^{D^B} is contained in S^D and lies in the kernel of the map $T : S^D \to S^{D^A}$ in Proposition 2.2.3. In particular, $S^D \geq S^{D^A} \oplus S^{D^B}$.*

Proof. For any $\sigma \in C_D \backslash C_{D^B}$, considered as a permutation of D^B, there exists some $p < q$ such that σ maps $x = (a_q, f(a_q))$ to $\sigma(x) = (a_p, f(a_p))$ with $f(a_p) \neq f(a_q)$. Since $y = (a_q, f(a_p)) \in D^B$, we have that x and y lie in row a_q but $\sigma(x)$ and $\sigma(y)$ lie in column $f(a_p)$. Therefore by Lemma 2.2.1, $C(D^B)\sigma R(D^B) = 0$.

Let U be a set of right coset representatives of $C_D \cap C_{D^B}$ in C_D and V a set of left coset representatives of $C_D \cap C_{D^B}$ in C_{D^B}. Then as in Proposition 2.2.3,

$$C(D^B)R(D^B) = C(D^B)\left(\sum_{\sigma\in U}\operatorname{sgn}(\sigma)\sigma\right)R(D) = \left(\sum_{\sigma\in V}\operatorname{sgn}(\sigma)\sigma\right)C(D)R(D).$$

It follows that $S^{D^B} \subset S^D$.

To see that it lies in the kernel of T, we need that

$$C(D^B)R(D^B)\sum_{\sigma \in Y} \sigma = C(D^B)\left(\sum_{\sigma \in X} \sigma\right) R(D^A) = 0.$$

We claim that $C(D^B)\sigma R(D^A) = 0$ for all $\sigma \in R_D = R_{D^B}$. Consider σ as a map from D^A to D^B. Suppose row i_2 of D^A has c boxes. Note that σ maps these c boxes into rows i_1 and i_2 of D^B, which lie in only $c-1$ columns. Thus σ maps two boxes from row i_2 of D^A into the same column of D^B, so Lemma 2.2.1 completes the proof. $\square$

We informally say that D "splits" into D^A and D^B, even though this split is not necessarily exact. Lemma 2.2.4 will be one of our key tools for decomposing Specht modules. Another such tool will be given in the next section.

2.3 Restricting Specht modules

In this section, we discuss the restriction of a Specht module from Σ_n to Σ_{n-1} (embedded so as to act on the first $n-1$ letters). This is motivated by the fact that there is a nice description of restrictions of irreducible representations, typically known as the *branching rule*.

We say a box of λ is a *corner box* if it has no box below it or to the right of it.

Proposition 2.3.1 (Branching rule)**.** *Let λ be a partition of n. Then*

$$S^\lambda|_{\Sigma_{n-1}} \cong \bigoplus_{\mu} S^\mu,$$

where μ ranges over all Young diagrams obtained from λ by removing a corner box.

Note that corner boxes have the property that no two lie in the same row or column, and no box of λ lies directly below one and directly to the right of another. In fact, we will show that any set of boxes with this property behaves in a similar fashion.

Definition. A subset U of boxes in a diagram D is a *transversal* if no two boxes of U lie in the same row or column. We say U is *special* if we can order the boxes of U as $(i_1, j_1), (i_2, j_2), \ldots$ such that for $p < q$, $(i_p, j_q) \notin D$.

Clearly the corner boxes of a Young diagram form a special transversal. We then have a similar branching rule result for any special transversal.

Proposition 2.3.2. *Let U be a special transversal of a diagram D with n boxes. Then*

$$S^D|_{\Sigma_{n-1}} \geq \bigoplus_{x \in U} S^{D \backslash \{x\}}.$$

Proof. By permuting rows and columns, we may assume that U consists of boxes (i, i) for $1 \leq i \leq u$ and that $(i, j) \notin D$ for $1 \leq i < j \leq u$. Write $D_i = D \backslash \{(i, i)\}$.

Recall that we interpret the basis elements of $\mathbf{C}[\Sigma_n]$ as tableaux of shape D with entries $1, \ldots, n$. For $1 \leq i \leq u$, let T_i be the set of tableaux of shape D such that n appears in box (i, i). By our choice of U, every term of $TC(D)R(D)$ for $T \in T_i$ contains the entry n in row at least i.

Let φ_i be the linear map on $\mathbf{C}[\Sigma_n]$ defined on a tableau T of shape D by $\varphi_i(T) = T$ if $T \in T_i$ and 0 otherwise. Since the action of Σ_{n-1} on tableaux in $\mathbf{C}[\Sigma_n]$ does not change the position of n, it follows that φ_i is a Σ_{n-1}-homomorphism.

Let V_i be the Σ_{n-1}-representation generated by $(T_i \cup T_{i+1} \cup \cdots \cup T_u)C(D)R(D)$ in $S^D|_{\Sigma_{n-1}}$. By above, we have that for $1 \leq i \leq u$, $V_{i+1} \subset \ker \varphi_i$. We then have that $\varphi_i(V_i)$ is generated by $\varphi_i(T_i C(D)R(D)) = T_i C(D_i)R(D_i)$, so it is naturally isomorphic to S^{D_i}. Therefore, the successive quotients of the filtration $0 \subset V_u \subset V_{u-1} \subset \cdots \subset V_1$ contain S^{D_u}, $S^{D_{u-1}}$, $\ldots$, S^{D_1} in succession. The result follows easily. $\square$

One might ask whether something similar to Proposition 2.3.2 can be said for restrictions of Schur modules. Indeed, if we consider $GL(N-1)$ as a subgroup of $GL(N)$ by acting on the first $N-1$ coordinates, then we can ask how $\mathscr{S}^D$ behaves as a restricted representation to $GL(N-1)$. Just as in the symmetric group case, there is a sort of branching rule for Young diagrams.

Definition. A *horizontal strip* of a Young diagram λ is a skew shape λ/μ containing at most one box in any column.

Proposition 2.3.3. *Let λ be a partition of n with at most N parts. Then*

$$\mathscr{S}^\lambda|_{GL(N-1)} \cong \bigoplus_\mu \mathscr{S}^\mu,$$

where μ ranges over all partitions obtained from λ by removing a horizontal strip.

It is possible to formulate a version of Proposition 2.3.2 for general Schur modules $\mathscr{S}^D$ (proved in a similar fashion), but because the description is more complicated, we will defer such a result until we specifically need it in Chapter 4.

Chapter 3

Matching polytopes

In this chapter we define the matching polytope of a graph. We also provide recurrences to facilitate the calculation of its volume. In the next chapter, we will use these results and the results of the previous chapter to draw a surprising connection between matching polytopes and Specht modules.

3.1 Definitions

We begin by defining the central object of this chapter.

Definition. Let $G = (V, E)$ be a graph. The *(fractional) matching polytope* M_G of G is the space of all nonnegative edge weightings $w\colon E \to \mathbf{R}_{\geq 0}$ such that for all $v \in V$,

$$\sum_{e \ni v} w(e) \leq 1.$$

This polytope, along with the related *perfect matching polytope* have been well studied in the field of combinatorial optimization (see, for instance, [13], [16]).

Note that M_G is a rational convex polytope of full dimension in $\mathbf{R}^n$, where $n = |E|$ is the number of edges in G. (Note also that this definition makes sense even if G has multiple edges.) The reason that M_G is called a matching polytope is due to the following definition.

Definition. A *matching* M of a graph G is a collection of edges of G such that no two edges of M share a vertex.

Given a matching M of G, let us write χ_M for the edge weighting of G defined by $\chi_M(e) = 1$ if $e \in M$ and 0 otherwise. Then clearly $\chi_M \in M_G$ for all matchings M. In fact, since χ_M is a vertex of the hypercube $[0,1]^{|E|}$, the χ_M are always vertices of M_G.

The following well known proposition characterizes when M_G is a lattice polytope. (See, for instance, [3].)

Proposition 3.1.1. *The matching polytope M_G is a lattice polytope if and only if G is bipartite. In this case, M_G is the convex hull of the χ_M, where M ranges over all matchings of G.*

It follows that when G is bipartite, the volume of M_G is an integer multiple of $\frac{1}{n!}$. Therefore, we let $V(G)$ be the *normalized volume* of M_G,

$$V(G) = n! \cdot \mathrm{vol}(M_G) \in \mathbf{Z}.$$

Recall that when we relate bipartite graphs to diagrams, it is important that we distinguish the two parts of our graph. Therefore, we will henceforth assume that all bipartite graphs are equipped with a bipartition of the vertices, that is, each vertex will be colored either black or white such that no two adjacent vertices have the same color. We will also assume that our graphs contain no isolated vertices. (Removing any isolated vertices will not change the matching polytope of the graph.)

Let $\mathscr{F}$ be the family of all finite (2-colored) forests. We will mostly be concerned with calculating $V(G)$ for $G \in \mathscr{F}$, so we will deal with this case first.

3.2 Calculating volumes

First, we have the following base case.

Proposition 3.2.1. *Let T_n be the star with n edges incident to a white center vertex. Then $V(T_n) = 1$.*

Proof. Let the edges have weights $w_1, \ldots, w_n$. Then M_G is defined by $0 \leq w_i$ for all i and $\sum_{i=1}^n w_i \leq 1$. This is the elementary n-simplex in $\mathbf{R}^n$, which has volume $\frac{1}{n!}$. $\square$

Next, we use the following recurrence to reduce to connected graphs.

Proposition 3.2.2. *Let G be a disjoint union of graphs $G_1 + G_2$, where G_1 has m edges and G_2 has $n - m$ edges. Then $V(G) = \binom{n}{m} V(G_1) V(G_2)$.*

Proof. Clearly $M_G = M_{G_1} \times M_{G_2}$. Hence

$$\begin{aligned} V(G) &= n! \cdot \mathrm{vol}(M_G) \\ &= n! \cdot \mathrm{vol}(M_{G_1}) \cdot \mathrm{vol}(M_{G_2}) \\ &= \frac{n!}{m! \cdot (m-n)!} \cdot V(G_1) \cdot V(G_2). \end{aligned} \qquad \square$$

We will now present a recurrence that will allow us to compute $V(G)$ for any $G \in \mathscr{F}$, which we call the leaf recurrence.

3.2.1 Leaf recurrence

Let H be a graph, and let v_1 and v_2 be distinct vertices of H. We construct three graphs G, G_1, and G_2 as follows. Let G be the graph obtained from H by adding pendant edges $\overline{v_1 v_1'}$ and $\overline{v_2 v_2'}$. (Then v_1' and v_2' are leaves in G.) Let G_1 be the graph obtained from H by adding a pendant edge $\overline{v_1 v_1'}$ and an edge $\overline{v_1 v_2}$, and let G_2 be obtained from H by adding a pendant edge $\overline{v_2 v_2'}$ and an edge $\overline{v_1 v_2}$. (See Figure 3-1.)

Proposition 3.2.3 (Leaf recurrence)**.** *Let G, G_1, and G_2 be as described above. Then $V(G) = V(G_1) + V(G_2)$.*

Proof. Consider the matching polytope M_G, and write w_i for the weight of $\overline{v_i v_i'}$. Let M_G^1 be the intersection of M_G with the halfspace $w_1 \geq w_2$, and let M_G^2 be the intersection of M_G with the halfspace $w_2 \geq w_1$. Clearly $\mathrm{vol}(M_G) = \mathrm{vol}(M_G^1) + \mathrm{vol}(M_G^2)$.

Consider the matching polytope M_{G_1}, and write z_1 for the weight of $\overline{v_1 v_1'}$ and z_2 for the weight of $\overline{v_1 v_2}$. Then for any $z \in M_{G_1}$, note that by letting $w_1 = z_1 + z_2$, $w_2 = z_2$,

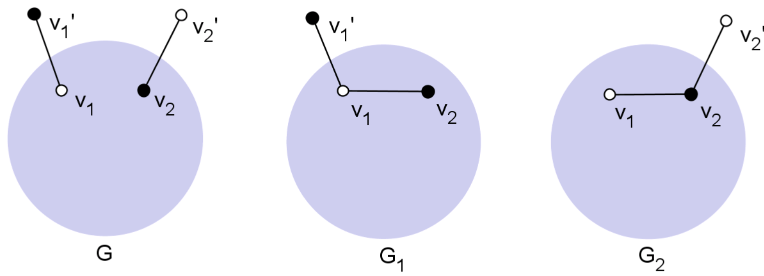


Figure 3-1: Three graphs as related by the leaf recurrence. They differ only in the edges marked; inside the circles the graphs are arbitrary as long as they are the same in all three cases.

and keeping all other weights the same, we obtain a point $f(z) \in M_G^1$. Moreover, f is a bijection from M_{G_1} to M_G^1: the inverse map is given by letting $z_1 = w_1 - w_2$, $z_2 = w_2$, and keeping all other weights the same. Since f is a volume-preserving linear transformation, it follows that $\mathrm{vol}(M_{G_1}) = \mathrm{vol}(M_G^1)$.

An analogous argument gives that $\mathrm{vol}(M_{G_2}) = \mathrm{vol}(M_G^2)$. The result follows. $\square$

Given any function f on forests, we will say that f *satisfies the leaf recurrence* if $f(G) = f(G_1) + f(G_2)$ for any three graphs G, G_1, and G_2 as described above.

We claim that the previous three propositions suffice to calculate $V(G)$ for any forest G.

Proposition 3.2.4. *There is a unique function $f\colon \mathscr{F} \to \mathbf{R}$ satisfying the following properties:*

1. *For the star T_n with n edges and white center vertex, $f(T_n) = 1$.*

2. *If G_1 and G_2 have m and $n - m$ edges, respectively, then $f(G_1 + G_2) = \binom{n}{m} f(G_1) f(G_2)$.*

3. *The function f satisfies the leaf recurrence.*

In this case, $f(G) = V(G)$, the normalized volume of the matching polytope of G.

Proof. Since we have seen that $V(G)$ satisfies these three properties, it suffices to show that for any forest G, one can determine $f(G)$ using these properties alone.

We induct first on n, the number of edges of G. By (2), we may then assume that G is connected and hence a tree. Choose a white vertex v_0 to be the root of G. We will then induct on s, the sum of the distances from all vertices to v_0. The base case is when $s = n$, which occurs only when G is T_n, in which case $f(G) = 1$ by (1).

Suppose $s > n$. Then G must have a leaf v_1' whose neighbor is $v_1 \neq v_0$. Let v_2 be the neighbor of v_1 closest to v_0, and let G' be the graph obtained from G by removing the edge $\overline{v_1 v_1'}$ and adding a pendant edge $\overline{v_2 v_2'}$. Then if H is the forest obtained from G by removing the edge $\overline{v_1 v_2}$ and adding a pendant edge $\overline{v_2 v_2'}$, we have that $f(H) = f(G) + f(G')$ by (3). Since H is disconnected, we can calculate $f(H)$ by induction. But $s(G') = s(G) - 1$, so we can determine $f(G')$ by induction as well. Thus we can determine $f(G) = f(H) - f(G')$, completing the proof. □

Note from the proof of Proposition 3.2.4 that even if we only stipulate that the three conditions hold only when all graphs involved have at most n edges, we still find that $f(G) = V(G)$ for all graphs with at most n edges.

3.2.2 Almost perfect matchings

Although the leaf recurrence suffices to calculate $V(G)$, it is not very efficient for calculations. Moreover, it does not give a *subtraction-free* rule for determining $V(G)$, in the sense that while it allows one to write $V(G)$ as a linear combination of multinomial coefficients, it does not give a nonnegative linear combination.

Therefore, we will introduce an alternative recurrence for calculating $V(G)$ for $G \in \mathscr{F}$. We first make the following definition.

Definition. We say that a matching M of G is *almost perfect* if every isolated edge of G lies in M and every non-leaf vertex of G is contained in an edge of M.

The importance of almost perfect matchings lies in the following proposition.

Proposition 3.2.5. *Let G be a bipartite graph, and let M be an almost perfect matching of G. Then*

$$V(G) = \sum_{e \in M} V(G \backslash e).$$

Proof. We claim that it is possible to partition M_G into cones C_e of volume $\frac{1}{n!}V(G\backslash e)$ for $e \in M$. Note that we may write $M_{G\backslash e} = M_G \cap \{w \mid w(e) = 0\}$. For $e \in M$, let C_e for the cone with vertex χ_M and base $M_{G\backslash e}$. Since C_e has height 1, it has volume $\frac{1}{n}\operatorname{vol}(M_{G\backslash e}) = \frac{1}{n!}V(G\backslash e)$, so it suffices to show that the C_e partition M_G (up to a measure zero set).

Let $w \in M_G$, and let e_0 be such that $w(e_0) = t$ is minimum among all $w(e)$ for $e \in M$. Let

$$w' = \frac{1}{1-t}(w - t \cdot \chi_M),$$

so that $w = t \cdot \chi_M + (1-t) \cdot w'$. Clearly $0 \le t \le 1$. By our choice of e_0, w' is a nonnegative weighting of $G\backslash e_0$. We claim that it lies in M_G (and hence in $M_{G\backslash e_0}$), which will imply that $w \in C_{e_0}$.

Note that although M_G is defined by an inequality for each vertex of G, the condition at a leaf that is not part of an isolated edge is redundant, for it is superseded by the condition at the adjacent vertex. Therefore, to check that $w' \in M_G$, it suffices to check that the sum of the weights of the edges incident to any non-leaf is at most 1 and that the weight on any isolated edge is at most 1. If v is a non-leaf of G, then

$$\sum_{e \ni v} w'(e) = \frac{1}{1-t}\left(\sum_{e \ni v} w(e) - t \cdot \sum_{e \ni v} \chi_M(e)\right) = \frac{1}{1-t}\left(\sum_{e \ni v} w(e) - t\right) \le 1.$$

Similarly, if e is an isolated edge of G, then $w'(e) = \frac{1}{1-t}(w(e) - t) \le 1$. It follows that $w' \in M_{G\backslash e_0}$, so $w \in C_{e_0}$. Therefore the cones C_e for $e \in M$ cover M_G.

We now show that the C_e have disjoint interiors. Suppose that w lies in the interior of C_{e_0}. Then we can write $w = t \cdot \chi_M + (1-t) \cdot w'$, where $0 < t < 1$ and w' lies in the interior of $M_{G\backslash e_0}$, so $w(e) = t + (1-t) \cdot w'(e)$ for $e \in M$. Since w' lies in the interior of $M_{G\backslash e_0}$, we have that $w'(e_0) = 0$, but $w'(e) > 0$ for all other $e \in M$. Therefore, e_0 is uniquely determined: $w(e_0)$ is the unique minimum among all $w(e)$ for $e \in M$. This proves the result. ∎

Not all graphs have almost perfect matchings, but every forest does.

Proposition 3.2.6. *Every forest has an almost perfect matching.*

Proof. It suffices to show that every rooted tree G with at least one edge has an almost perfect matching M such that the root lies in an edge of M. We induct on the number of edges of G, with the base case being trivial. Choose any edge e incident to the root, and consider the forest G' obtained from G by removing both endpoints of e as well as any edge incident to either endpoint. Root each component of G' at the vertex that was closest to e in G. By induction, any component with at least one edge has an almost perfect matching containing an edge that contains the root. It is then easy to check that the union of these matchings together with e is an almost perfect matching of G. □

It follows that we can use Proposition 3.2.5 to recursively compute $V(G)$ for any $G \in \mathscr{F}$. One way to express this is to say that $V(G)$ counts the number of standard labelings of G in the following sense.

Definition. Fix an almost perfect matching $M(G)$ for every $G \in \mathscr{F}$. For any $G \in \mathscr{F}$, we say that an edge labeling $z\colon E \to \mathbf{N}$ is *standard* if z is a bijection between E and $[n] = \{1, \ldots, n\}$ such that $z^{-1}(n) \in M(G)$ and, if $n > 1$, $z|_{G \backslash z^{-1}(n)}$ is a standard labeling of $G \backslash z^{-1}(n)$.

The following proposition is then immediate.

Proposition 3.2.7. *For any $G \in \mathscr{F}$, $V(G)$ is the number of standard labelings of G.*

Proof. Both $V(G)$ and the number of standard labelings of G satisfy the recurrence in Proposition 3.2.5. □

While Proposition 3.2.5 has the advantage of being a subtraction-free rule, it has the disadvantage that unlike the leaf recurrence, it is not homogeneous. In other words, here $V(G)$ is written in terms of $V(H)$ where H does not have the same number of edges as G.

As it happens, we will use both Propositions 3.2.3 and 3.2.5 to deduce the main result of the following chapter.

3.2.3 Other recurrences

While it is not important for the rest of this thesis, we also present recurrences that will allow us to compute $V(G)$ for any graph (bipartite or not). Since we have already described how to calculate the volume for forests, the main idea will be to get rid of cycles. For the rest of this section, we allow graphs with multiple edges.

The leaf recurrence is a special case of the following more general volume relation. Geometrically, it reflects the fact that if we project M_G onto a hyperplane, the total projected volume of the "upper" facets is the same as that the "lower" facets.

Proposition 3.2.8. *Let G be a graph with edge set E, and let $w : E \to \mathbf{R}$ be an edge weighting such that for every vertex v of G that is either a nonleaf or lies in a component with only two vertices, $\sum_{e \ni v} w(e) = 0$. Then*

$$\sum_{e \in E} w(e) V(G \backslash e) = 0.$$

Proof. Let E^+ be the set of edges on which w is positive and E^- the set on which w is negative. For each $e^+ \in E^+$ and $e^- \in E^-$, let $M_{e^-e^+}$ be the set of all $z \in M_G$ such that $z(e^-) = 0$ and $|z(e^+)/w(e^+)|$ is minimum among all edges in E^+. Similarly define $M_{e^+e^-}$. Clearly for fixed e^-, the volume of $M_{G \backslash e^-}$ is the sum of the volumes of the $M_{e^-e^+}$, and likewise for $M_{e^+e^-}$.

We claim that the linear map that sends $z \mapsto z'$, where

$$z'(e) = z(e) - \frac{z(e^+)}{w(e^+)} \cdot w(e),$$

maps $M_{e^-e^+}$ exactly onto $M_{e^+e^-}$. To see this, first note that the sum of the weights around each nonleaf is unchanged by the condition on w. Also the weight on each edge of E^- increases. To see that $z'(f^+)$ is always nonnegative for $f^+ \in E^+$, note that whenever $w(e) \neq 0$,

$$\frac{z'(e)}{w(e)} = \frac{z(e)}{w(e)} - \frac{z(e^+)}{w(e^+)}.$$

For $e = f^+$, this is always nonnegative by the definition of $M_{e^-e^+}$. Clearly $z'(e^+) = 0$.

Finally, $|z'(e^-)/w(e^-)|$ is minimum among edges of E^-, since it was zero before, and all such quantities increase by the fixed amount $z(e^+)/w(e^+)$. This proves the claim.

The map $z \mapsto z'$ is an invertible linear map (from $\mathbf{R}^{|E \setminus e^-|}$ to $\mathbf{R}^{|E \setminus e^+|}$) of determinant $|w(e^-)/w(e^+)|$. Therefore

$$w(e^-) \cdot \operatorname{vol} M_{e^-e^+} + w(e^+) \cdot \operatorname{vol} M_{e^+e^-} = 0.$$

Summing over all $e^+ \in E^+$ and $e^- \in E^-$ gives the result. □

Suppose G has pendant edges e_1 and e_2 both connected to an edge e. Then letting $w(e_1) = w(e_2) = 1$ and $w(e) = -1$ in Proposition 3.2.8 gives the leaf recurrence.

Different weightings give us different relations, including the following.

Proposition 3.2.9. *Let G be a graph containing an even cycle with edges e_1, e_2, …, e_{2k} in order. Then*

$$\sum_{i \text{ odd}} V(G \backslash e_i) = \sum_{i \text{ even}} V(G \backslash e_i).$$

Proof. Let $w(e_i) = (-1)^i$, $w(e) = 0$ otherwise, and apply Proposition 3.2.8. □

Suppose H is a graph with an even cycle C (of length at least 4) containing edge e. If we use Proposition 3.2.9 where G is the graph obtained from H by duplicating edge e, then we find that $V(H)$ can be written as a sum of $\pm V(H')$, where H' is obtained from H by duplicating edge e and removing another edge in the cycle. If the edge removed was a multiple edge, we can repeat the procedure, each time increasing the multiplicity of edge e. Eventually, we will be able to write $V(H)$ as the sum of $\pm V(H')$ where each H' does not contain the cycle C.

Repeating this procedure for each cycle of H, we can write $V(H)$ as the sum of $\pm V(H')$ where each H' is a forest (though possibly with multiple edges). But we already know how to calculate $V(H')$ for such graphs (H' still has an almost perfect matching, or alternatively, the procedure of Proposition 3.2.4 still applies). Therefore Proposition 3.2.9 allows us to compute $V(G)$ for any bipartite (multi)graph.

We can also develop a recurrence to compute $V(G)$ when G is not a bipartite graph. Recall that in this case, the vertices of M_G are not lattice points. (Since G

contains an odd cycle, weighting each edge in the cycle by $\frac{1}{2}$ gives a noninteger vertex of M_G.) However, we can still compute $V(G) = n! \cdot \operatorname{vol} M_G$ even if it is not necessarily an integer.

Proposition 3.2.10. *Let G be a graph containing an odd cycle with edges e_1, …, e_{2k+1} in order, as well as a pendant edge e' adjacent to e_1 and e_{2k+1}. Then*

$$\sum_{i \text{ odd}} V(G \backslash e_i) = \sum_{i \text{ even}} V(G \backslash e_i) + 2V(G \backslash e').$$

Proof. Let $w(e_i) = (-1)^i$, $w(e') = 2$, and $w(e) = 0$ otherwise. Then apply Proposition 3.2.8. $\square$

If H is a graph containing an odd cycle C, we may let G be a graph obtained from H by adding a pendant edge e' to one vertex of C and then apply Proposition 3.2.10. Then $H = G \backslash e'$, so we may write $2V(H)$ as a sum of $\pm V(H')$, where H' differs from H by removing an edge of C and adding e'. In particular, each H' no longer contains the cycle C. Continuing in this manner for all odd cycles, we can write some multiple of $V(H)$ as a sum of $\pm V(H')$, where each H' is bipartite. It follows that Proposition 3.2.10 allows us to compute the volume of the matching polytope of any graph.

In the next chapter, we will apply the results above to finding the dimension of the Specht module S^G for $G \in \mathscr{F}$.

Chapter 4

Forests

The main result of this chapter will be to show that the normalized volume of the matching polytope of a forest G is equal to the dimension of its Specht module. We will also describe some properties of the Schur function s_G and define a notion of semistandard tableaux for forests.

4.1 Main result

Recall that to each diagram D of boxes, we associate a 2-colored bipartite graph G as follows. Each white vertex of G is a row of D, each black vertex is a column of D, and a white vertex is connected to a black vertex if there is a box of D in the intersection of the corrosponding row and column.

We can now prove the following somewhat surprising relation between matching polytopes and Specht modules of forests.

Theorem 4.1.1. *For all $G \in \mathscr{F}$, $\dim S^G = V(G)$.*

The main idea is to relate the results of the previous two chapters. First, we relate the leaf recurrence with splitting Specht modules in the following way.

Proposition 4.1.2. *Let G, G_1, and G_2 be three forests as appearing in Proposition 3.2.3 (the leaf recurrence). Then $S^G \geq S^{G_1} \oplus S^{G_2}$.*

Proof. The diagram of G splits into G_1 and G_2 as in Lemma 2.2.4. (Take the boxes (i_1, j_1) and (i_2, j_2) to be the two pendant edges of interest in G.) $\square$

Second, we relate the almost perfect matching recurrence for forests to the special transversal recurrence for Specht modules. Clearly a transversal of a diagram is equivalent to a matching of the corresponding graph.

Proposition 4.1.3. *Let U be a transversal of a diagram D, and let M be the corresponding matching in the graph $G = G(D)$. Let E be the subdiagram of D given by the intersection of the rows and columns of D containing boxes in U. Then the following are equivalent:*

1. *U is a special transversal of D.*

2. *U is the unique transversal in E of size $|U|$.*

3. *There does not exist a cycle in G half of whose edges lie in M.*

Proof. For the first equivalence, suppose U is the unique transversal in E of size $|U|$. We may assume that U consists of the boxes (i, i) for $1 \leq i \leq u$. Consider the directed graph H on $[u]$ with an edge from i to j if $(i, j) \in E \backslash U$. In fact H is acyclic: if $(i_0, i_1, \ldots, i_s = i_0)$ formed a cycle in H, then we could replace (i_j, i_j) in U by (i_j, i_{j+1}) for $0 \leq j < s$ and obtain another transversal in E of size $|U|$. It follows that we can reorder the vertices of H such that there is an edge from i to j only if $i > j$. This gives us an ordering of the rows and columns of E that shows that U is special. The other direction is easy.

For the second equivalence, suppose that there exists a cycle in G half of whose edges lie in M. Replacing those edges in M with the other edges in the cycle gives another matching M', and the corresponding transversal U' lies in the same rows and columns of D as U. Conversely, suppose U and U' are both transversals of E of size $|U|$. If M and M' are the corresponding matchings, then $(M \backslash M') \cup (M' \backslash M)$ consists of two disjoint matchings on the same set of vertices. It therefore has the same number of edges as vertices, so it contains a cycle. Since both M and M' are matchings, exactly half of the edges in the cycle must lie in each of M and M'. $\square$

Since a forest has no cycles, the third condition is automatically satisfied, meaning that every matching of a forest is special. We can therefore deduce the following.

Proposition 4.1.4. *For all $G \in \mathscr{F}$, $\dim S^G \geq V(G)$.*

Proof. We induct on the number of edges in G. Let M be an almost perfect matching of G. Since G is a forest, M is special, so by Proposition 2.3.2, the induction hypothesis, and Proposition 3.2.5,

$$\dim S^G \geq \sum_{e \in M} \dim S^{G \backslash e} \geq \sum_{e \in M} V(G \backslash e) = V(G). \qquad \square$$

We can now combine these results to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. We wish to show that for any $G \in \mathscr{F}$, $\dim S^G = V(G)$. It suffices to show that $\dim S^G$ satisfies the three conditions given in Proposition 3.2.4. We induct on n, the number of edges of G, noting as before that if we prove the conditions for $n \leq k$, then $\dim S^G = V(G)$ for all graphs with at most k edges. Let us then assume that we have proven the claim for all graphs with fewer than n edges.

The first condition states that if T_n is the tree with n edges and white center vertex, then $\dim S^{T_n} = 1$. But the diagram of T_n is a row of n boxes, and its Specht module is the trivial representation, which has dimension 1. (This also proves the base case when $n = 1$.)

The second condition states that if G_1 and G_2 have m and $n-m$ edges, respectively, then the disjoint union $G_1 + G_2$ satisfies $\dim S^{G_1+G_2} = \binom{n}{m} \cdot \dim S^{G_1} \cdot \dim S^{G_2}$. We have that

$$\begin{aligned} \dim S^{G_1+G_2} &= \dim \operatorname{Ind}_{\Sigma_m \times \Sigma_{n-m}}^{\Sigma_n} (S^{G_1} \otimes S^{G_2}) \\ &= [\Sigma_n : \Sigma_m \times \Sigma_{n-m}] \cdot \dim(S^{G_1} \otimes S^{G_2}) \\ &= \binom{n}{m} \cdot \dim S^{G_1} \cdot \dim S^{G_2}. \end{aligned}$$

In particular, this implies by induction that $\dim S^G = V(G)$ for any disconnected graph G with n edges.

The third and final condition states that $\dim S^G$ satisfies the leaf recurrence. In other words, we need to show that for $G, G_1, G_2 \in \mathscr{F}$ related as in Proposition 3.2.3, $\dim S^G = \dim S^{G_1} + \dim S^{G_2}$. Since G is disconnected, we have from above that $\dim S^G = V(G)$. But by Propositions 4.1.2, 4.1.4, and 3.2.3, we have

$$V(G) = \dim S^G \geq \dim S^{G_1} + \dim S^{G_2} \geq V(G_1) + V(G_2) = V(G).$$

Therefore we must have equality everywhere. The result follows immediately. $\square$

Note that equality must hold everywhere in the proof above, and therefore we must also have equality in Propositions 4.1.2 and 4.1.4. This gives us the following two corollaries.

Corollary 4.1.5. *Let $G, G_1, G_2 \in \mathscr{F}$ be related as in Proposition 3.2.3 (the leaf recurrence). Then*

$$S^G \cong S^{G_1} \oplus S^{G_2}.$$

Corollary 4.1.6. *Let $G \in \mathscr{F}$, and let M be an almost perfect matching of G. Then*

$$S^G|_{\Sigma_{n-1}} \cong \bigoplus_{e \in M} S^{G \setminus e}.$$

In particular, Corollary 4.1.6 shows that any almost perfect matching in a forest plays a similar role as the corner boxes in a Young diagram.

It is perhaps worth noting that Reiner and Shimozono's result on percentage-avoiding shapes does not cover most forests.

Proposition 4.1.7. *Let G be any graph containing as an induced subgraph the tree formed by joining three paths of length three at a vertex. Then G is not %-avoiding.*

Proof. Suppose that the ith path has edges (r_i, c), (r_i, d_i), and (s_i, d_i), where $r_1 < r_2 < r_3$. In order to avoid a %-shape between (r_i, d_i) and (r_j, d_j), we must have $d_1 < d_2 < d_3$.

Suppose $d_2 < c$. Then (s_2, d_2) must form a %-shape with either (r_1, c) (if $s_2 > r_1$) or (r_1, d_1) (if $s_2 < r_1$). The case $d_2 > c$ is similar. $\square$

4.2 Schur functions

Recall that the Schur function s_G is defined by $\operatorname{ch}\chi^G$, where χ^G is the character of S^G and ch is the Frobenius characteristic map that sends χ^λ to s_λ. In the case when $G \in \mathscr{F}$, we can now give an alternate characterization of s_G.

Proposition 4.2.1. *There is a unique function $s\colon \mathscr{F} \to \Lambda_{\mathbf{Z}}$ that satisfies the following properties:*

1. *For the star T_n with n edges and white center vertex, $s(T_n) = h_n$, the nth complete homogeneous symmetric function.*

2. *For $G_1, G_2 \in \mathscr{F}$, $s(G_1 + G_2) = s(G_1)s(G_2)$.*

3. *The function s satisfies the leaf recurrence.*

In this case, $s(G) = s_G$, the Schur function associated to G.

Proof. The proof of uniqueness is identical to that of Proposition 3.2.4, so it suffices to check that the map $G \mapsto s_G$ satisfies these properties. The first follows because the Schur function corresponding to the partition (n) is h_n. The second follows because the analogous statement holds for partitions. The third follows immediately from Corollary 4.1.5. $\square$

Indeed, we can even say something a little bit stronger: the map s is universal in the following sense.

Theorem 4.2.2. *Let R be a commutative ring with unit and $\bar{f}\colon \{T_n \mid n > 0\} \to R$ any function. Then $\bar{f}$ can be uniquely extended to a function $f\colon \mathscr{F} \to R$ that satisfies:*

1. *For $G_1, G_2 \in \mathscr{F}$, $f(G_1 + G_2) = f(G_1)f(G_2)$.*

2. *The function f satisfies the leaf recurrence.*

Moreover, f factors uniquely as $\varphi \circ s$, where $s\colon G \mapsto s_G$ and $\varphi\colon \Lambda_{\mathbf{Z}} \to R$ is a ring homomorphism.

Proof. Recall that the h_n are algebraically independent, so that $\Lambda_{\mathbf{Z}} = \mathbf{Z}[h_1, h_2, \ldots]$. Setting $\varphi(h_n) = \bar{f}(T_n)$ defines φ uniquely, and the claim follows easily from Proposition 4.2.1. □

For example, recall that the map $G \mapsto \operatorname{vol} M_G$ satisfies the conditions of Theorem 4.2.2. It follows that this map corresponds to a ring homomorphism $\Lambda_{\mathbf{Z}} \to \mathbf{Q}$. Indeed, this is the map known as *exponential specialization*, which maps a homogeneous function $f \in \Lambda_{\mathbf{Z}}$ of degree n to the coefficient $\frac{1}{n!}[x_1 x_2 \ldots x_n]f$. (See, for instance, [23].)

As another example, we may take the map $\Lambda_{\mathbf{Z}} \to \mathbf{Q}[N]$ that sends a symmetric function $f(x_1, x_2, \ldots)$ to its evaluation when N of the variables equal 1 and the rest are 0, sometimes called its *principal specialization*. We find that this has the following combinatorial interpretation.

Proposition 4.2.3. *Let $G \in \mathscr{F}$. For any nonnegative integer N, let $m_G(N)$ be the number of nonnegative integer edge labelings $w\colon E \to \mathbf{Z}$ such that for all $v \in V$,*

$$\sum_{e \ni v} w(e) \leq \begin{cases} N-1, & \text{if } v \text{ is white, and} \\ N - \deg(v), & \text{if } v \text{ is black.} \end{cases}$$

Then $m_G(N)$ is a polynomial in N, the map $G \mapsto m_G$ satisfies the conditions of Theorem 4.2.2, and $m_G(N) = s_G(\underbrace{1, 1, \ldots, 1}_{N})$.

Proof. First suppose $G = T_n$. Then $m_G(N)$ is the number of n-tuples of nonnegative integers summing to at most $N-1$, and a simple counting argument shows that this is $\binom{n+N-1}{n} = h_n(\underbrace{1, \ldots, 1}_{N})$.

It suffices to check that the map $G \mapsto m_G$ satisfies the conditions of Theorem 4.2.2. The first condition is obvious. For the second, we mimic the proof of Proposition 3.2.3. Let G, G_1, G_2, v_1, v_1', v_2, and v_2' be as before, and let us assume without loss of generality that v_1 is white and v_2 is black. Let w be a suitable edge weighting for G with $w_i = w(\overline{v_i v_i'})$. If $w_1 \geq w_2$, then let z be the weighting of G_1 such that $z(\overline{v_1 v_2}) = w_2$, $z(\overline{v_1 v_1'}) = w_1 - w_2$, and $z(e) = w(e)$ for all other edges; if $w_1 < w_2$, let z' be the weighting of G_2 such that $z'(\overline{v_1v_2}) = w_1$, $z'(\overline{v_2v_2'}) = w_2 - w_1 - 1$, and $z'(e) = w(e)$ for all other edges. It is simple to check that this gives a bijection between suitable edge weightings for G and suitable edge weightings for either G_1 or G_2. The result follows. $\square$

Therefore the principal specialization gives a polynomial that counts lattice points in a way that is similar to the Ehrhart polynomial of the matching polytope.

We can also give a representation-theoretic interpretation of $m_G(N)$ in terms of Schur modules.

Proposition 4.2.4. *Let $G \in \mathscr{F}$. Then $m_G(N) = \dim \mathscr{S}^G$.*

Proof. Follows from Proposition 4.2.3, since $\dim \mathscr{S}^D = s_D(\underbrace{1, 1, \ldots, 1}_{N})$. $\square$

4.3 Semistandard tableaux

Recall that for Specht modules, Corollary 4.1.6 gives a description of the restriction of S^G from Σ_n to Σ_{n-1} by identifying a set of edges that act analogously to corner boxes of a Young diagram. We shall now obtain a similar result for the restriction of $\mathscr{S}^G$ from $GL(N)$ to $GL(N-1)$ (where $GL(N-1) \subset GL(N)$ acts on the first $N-1$ coordinates). This amounts to finding certain subsets of edges that act analogously to horizontal strips of a Young diagram. This will allow us to define a notion of semistandard tableaux for forests.

Let D be the diagram of a forest, and fix once and for all a transversal corresponding to an almost perfect matching U. Let us assume that U consists of the boxes (i, i) for $1 \leq i \leq u$ and that $(i, j) \notin D$ for $1 \leq i < j \leq u$ (so that the set U is also equipped with an ordering). We will say that such a diagram is in *standard form.*

Lemma 4.3.1. *Let D be the diagram of a forest with transversal U as above. Then the diagram D' obtained by removing column u from D has a transversal U' corresponding to an almost perfect matching that contains a box from each of the first $u-1$ columns of D' (and possibly others).*

Proof. Let x_0 be the box (u, u), and choose distinct boxes x_i and y_i recursively such that y_i lies in the same row as x_i (if such a box exists), and x_{i+1} lies in the same column as y_i and in U (if such a box exists). Since D is the diagram of a forest, this must terminate. Then simply replace the set of all x_i in U with the set of all y_i to get U'. □

In the definition below, we will write D' for the diagram obtained from D by removing column u, and we associate to it the transversal U' as in the lemma. Also, let D'' be the diagram obtained from D by removing row and column u and associate to it the transversal $U'' = U \backslash \{(u, u)\}$.

Definition. A subset $Y \subset D$ is called a *horizontal strip* (with respect to U) if either $Y = \{(u, u)\} \cup Y'$, where Y' is a horizontal strip of D' with respect to U', or $Y = Y''$, where Y'' is a horizontal strip of D'' with respect to U''. (The empty set is a horizontal strip for any diagram D.)

Here we have abused notation slightly: D' may not be in standard form with respect to U', but it is equivalent to some diagram E' that is in standard form with respect to the image of U'. By a horizontal strip of D' we mean the boxes of D' that correspond to a horizontal strip of E'.

It is easy to check that the horizontal strips of size 1 exactly correspond to elements of U, just as for partitions, a horizontal strip of length 1 is a corner box. It is also easy to check that a horizontal strip can have at most one box in any column but can have more than one box in a row.

Note that the set of horizontal strips of D can vary greatly according to the choices and orderings of almost perfect matchings. We will therefore fix once and for all a single such choice for each diagram D, yielding a single set $\mathcal{Y}(D)$ of horizontal strips.

Although the definition of a horizontal strip is a bit unusual, its value lies in the following result, the analogue of Proposition 3.2.5.

Proposition 4.3.2. *Let D be the diagram of a forest and $N > 1$ a positive integer.*

Then

$$m_D(N) = \sum_{Y \in \mathcal{Y}(D)} m_{D \setminus Y}(N-1).$$

Proof. Let $W_D(N)$ be the set of weightings of D as enumerated by $m_D(N)$ in Proposition 4.2.3. In other words, $W_D(N)$ is the number of labelings of the boxes of D with nonnegative integers such that the sum of each row is at most $N-1$ and the sum of each column is at most N minus the number of boxes in that column.

Note that the conditions on any row or column after the first u are superfluous: since the diagonal corresponds to an almost perfect matching, any row after the first u contains at most one box, and any such box lies in one of the first u columns. Then the condition from the row states that this box is labeled at most $N-1$, which is automatically implied by the condition on its column. (The same argument holds for columns after the first u.)

We will construct a bijection $f\colon \bigcup_{Y \in \mathcal{Y}(D)} W_{D \setminus Y}(N-1) \to W_D(N)$ inductively on the number of boxes of D. This bijection will satisfy the following properties: for $w \in W_{D \setminus Y}(N-1)$,

- $f(w)$ is obtained from w by inserting boxes labeled 0 at the positions of Y and increasing the labels of certain other boxes of $D \setminus Y$ by 1;

- each of the first u columns will have either a box inserted or a box whose label is increased (not both);

- no row will contain two boxes with labels that are increased.

In particular, this bijection will increase the sum of any row or column by either 0 or 1, and, more specifically, the sum of any of the first u columns will either increase by 1, or else the sum will stay the same but the column will gain a box. Note that this will imply that the image of f is contained in $W_D(N)$.

For the forward direction, let $w \in W_{D \setminus Y}(N-1)$. If $(u,u) \in Y$, then let z be the labeling of D such that $z(u,u) = 0$, $z|_{D'} = f(w|_{D' \setminus Y})$ by induction, and z and w agree elsewhere. If $(u,u) \notin Y$, then let z be such that $z(u,u) = w(u,u) + 1$,

$z|_{D''} = f(w|_{D''\setminus Y})$, and z and w agree elsewhere. It is easy to check by induction that all the conditions above are satisfied.

It remains to verify that f is a bijection. Choose $z \in W_D(N)$, and suppose $z(u,u) = 0$. Then if $f(w) = z$, then w must be unique. Indeed, first note that we must have $(u,u) \in Y$. Since $z|_{D'} \in W_{D'}(N)$, by induction we can find $f^{-1}(z|_{D'}) \in W_{D'\setminus Y'}(N-1)$ for some horizontal strip Y' of D'. Then we must have $Y = \{(u,u)\} \cup Y'$, $w|_{D'\setminus Y'} = f^{-1}(z|_{D'})$, and w and z agree elsewhere on $D\setminus Y$, so w is uniquely determined. We then need to verify that such a w exists, that is, that it lies in $W_{D\setminus Y}(N-1)$. Since $w|_{D'\setminus Y'} \in w_{D'\setminus Y'}(N-1)$, the sum of any of the first u rows is at most $N-2$. Similarly, the sum of any of the first $u-1$ columns is at most $N-1$ minus the length of the column. This also holds for column u, for the sum of this column in z was at most N minus the length, and while the sum in w is the same, we removed a box. Thus $w = f^{-1}(z)$ is well-defined and unique in this case.

If $z(u,u) \neq 0$, then we similarly find that we must have $(u,u) \notin Y$ and $f^{-1}(z|_{D''}) \in W_{D''\setminus Y''}(N-1)$ for $Y'' = Y$, so w is determined by $w|_{D''\setminus Y''} = f^{-1}(z|_{D''})$, $w(u,u) = z(u,u)-1$, and w and z agree elsewhere. We again need to check that $w \in W_{D\setminus Y}(N-1)$. Since $w|_{D''\setminus Y''} \in W_{D''\setminus Y''}(N-1)$, the sum of any of the first u rows is at most $N-2$ (even row u, for it was at most $N-1$ and we subtracted 1 from (u,u)). Similarly, the sum of any of the first u columns is at most $N-1$ minus the length of the column, for in each of these columns we either decreased a label by 1 or deleted a box (by the properties above). Thus $w = f^{-1}(z)$ is well-defined and unique in this case as well. The result now follows from Proposition 4.2.3. $\square$

We may now use this result to prove the following analogue of Corollary 4.1.6.

Theorem 4.3.3. *Let D be the diagram of a forest and $\mathscr{S}^D$ the corresponding $GL(N)$-module. Then*

$$\mathscr{S}^D|_{GL(N-1)} \cong \bigoplus_{Y \in \mathcal{Y}(D)} \mathscr{S}^{D\setminus Y}$$

as $GL(N-1)$-modules.

Proof. By Proposition 4.3.2, we have that the dimension of both sides of this equation are equal. It therefore suffices to show that the left side contains the right side as a submodule.

For a horizontal strip Y, let T_Y be the set of all tableaux T with boxes in Y labeled N and all other boxes labeled at most $N-1$. Define the map φ_Y by $\varphi_Y(T) = T$ if $T \in T_Y$ and 0 otherwise. We may identify φ_Y with a linear map on $V^{\otimes n}$. Since $GL(N-1)$ stabilizes $e_1, \ldots, e_{N-1}$ and fixes e_N, φ_Y is a $GL(N-1)$-homomorphism.

We construct a linear order $\prec$ on $\mathcal{Y}(D)$ inductively on the number of boxes of D as follows:

1. If $(u,u) \in Y$ and $(u,u) \notin Z$, then $Z \prec Y$;

2. If $(u,u) \notin Y, Z$, then $Y'' = Y, Z'' = Z \in \mathcal{Y}(D'')$, so write $Z \prec Y$ if and only if $Z'' \prec Y''$;

3. If $(u,u) \in Y, Z$, then $Y' = Y \backslash \{(u,u)\}, Z' = Z \backslash \{(u,u)\} \in \mathcal{Y}(D')$, so write $Z \prec Y$ if and only if $Z' \prec Y'$.

Let $T \in T_Y$. We claim that $\prec$ has the property that if $Z \prec Y$, then $TC(D)R(D)$ does not contain a term in T_Z. In other words, if $Z \prec Y$, then $Tqp \notin T_Z$ for $q \in C_D$, $p \in R_D$. To see this, we check that it holds for the three defining conditions above.

For (1), if $(u,u) \in Y$, then Tqp contains a box labeled N in either row u or column u. Hence by the definition of horizontal strips, it cannot lie in T_Z if $(u,u) \notin Z$.

For (2), note that if $(u,u) \notin Y, Z$, then Y and Z do not contain any boxes in row u or column u. Thus we may assume that q and p do not affect row u and column u. If $Tqp \in T_Z$, then some term of $T|_{D''}\bar{q}\bar{p}$ lies in $T_{Z''}$, where $\bar{q}$ and $\bar{p}$ correspond to q and p in $C_{D''}$ and $R_{D''}$. Then we cannot have $Z'' \prec Y''$ and hence neither can we have $Z \prec Y$.

For (3), suppose $Tqp \in T_Z$. Note that Y and Z contain only one box in column u, namely (u,u). Then q must send box (u,u) to itself, and p sends (u,u) to another box (u,v). Let σ be the transposition in R_D switching (u,u) and (u,v). Then by replacing p by $p\sigma$, we may assume that neither q nor p affects column u. Then restricting to D' as in the case above shows this case as well.

It follows that if $T \in T_Y$ and $Z \prec Y$, then $\varphi_Z(TC(D)R(D)) = 0$. Note that the analysis above also shows that if $Tqp \in T_Y$, then q acts as the identity on Y, so that we may associate q to an element of $C_{D\backslash Y}$. From this, it follows that $\varphi_Y(TC(D)R(D))$ is a scalar multiple of $TC(D\backslash Y)R(D\backslash Y)$ (by the order of the subgroup of R_D that stabilizes Y).

With this, we are ready to prove the result. Index the horizontal strips such that $Y_1 \prec Y_2 \prec \cdots \prec Y_m$, and let V_i be the subrepresentation of $\mathscr{S}^D|_{GL(N-1)}$ generated by $(T_{Y_i} \cup T_{Y_{i+1}} \cup \cdots \cup T_{Y_m})C(D)R(D)$. By the discussion above, $V_{i+1} \subset \ker \varphi_{Y_i}$, while $\varphi_{Y_i}(V_i)$ is generated by $\varphi_{Y_i}(TC(D)R(D))$ for $T \in T_{Y_i}$ and hence is naturally isomorphic to $\mathscr{S}^{D\backslash Y_i}$. Therefore the successive quotients of $0 \subset V_m \subset V_{m-1} \subset \cdots \subset V_1$ contain $\mathscr{S}^{D\backslash Y_m}$, $\mathscr{S}^{D\backslash Y_{m-1}}$, $\ldots$, $\mathscr{S}^{D\backslash Y_1}$ in succession. It follows from the complete reducibility of $GL(N-1)$-representations that we have the desired inclusion, and the result follows. $\square$

In fact, this shows that if for each $Y \in \mathcal{Y}(D)$, there exists a set of tableaux $SS(D\backslash Y, N-1)$ such that $SS(D\backslash Y, N-1)C(D\backslash Y)R(D\backslash Y)$ forms a basis of the $GL(N-1)$-module $\mathscr{S}^{D\backslash Y}$, then we can construct a basis of the $GL(N)$-module $\mathscr{S}^D$ simply by adding boxes labeled N to the tableaux in $SS(D\backslash Y, N-1)$ along Y for all Y. This leads us to the following definition.

Definition. Let D be the diagram of a forest, and let T be a tableau of shape D with entries at most N. We say that T is *semistandard* if the boxes labeled by N in T form a horizontal strip in $\mathcal{Y}(D)$, and the tableau formed from T by removing all boxes labeled N is also semistandard. (The empty tableau is semistandard by default.) We say that T is *standard* if it is semistandard and contains only the entries $1, \ldots, n$, each exactly once.

(Note that since horizontal strips of length 1 on D precisely correspond to edges in an almost perfect matching of the graph of G, this definition agrees with the definition of standard labelings given in Chapter 3.)

It is easy to see that a semistandard tableau is equivalent to a sequence of diagrams $\emptyset = D^{(0)} \subset D^{(1)} \subset \cdots \subset D^{(N)} = D$ such that $D^{(i)}\backslash D^{(i-1)} \in \mathcal{Y}(D^{(i)})$ for $1 \le i \le N$.

We denote the set of semistandard tableaux of shape D with entries at most N by $SS(D, N)$ and the set of all semistandard tableaux of shape D by $SS(D)$.

Proposition 4.3.4. *Let D be the diagram of a forest. Then*

$$s_D(x_1, x_2, \dots) = \sum_{T \in SS(D)} x_1^{\alpha_1} x_2^{\alpha_2} \cdots ,$$

where α_i is the number of occurrences of i in T. In particular, $\dim(S^D) = V(D)$ *is the number of standard tableaux of shape D.*

Proof. From the discussion after the proof of Theorem 4.3.3, the $TC(D)R(D)$, for $T \in SS(D, N)$, form a basis for $\mathscr{S}^D$. Thus, since $s_D(x_1, \dots, x_N)$ is the character of $\mathscr{S}^D$, and $\operatorname{diag}(x_1, \dots, x_N)$ acts on $TC(D)R(D)$ by multiplication by $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, the result follows easily. $\square$

This therefore provides a combinatorial description of the coefficients of s_D.

In the next chapter, we discuss Schubert varieties for general diagrams and show how Theorem 4.1.1 implies a special case of Conjecture 1.

Chapter 5

Schubert varieties

In this chapter, we describe some properties of Schubert varieties for general diagrams. We will also consider the special case in which the diagram is the complement of a forest, and we will prove our main conjecture in this case. We will also prove the weaker Conjecture 2 for row convex diagrams and Rothe diagrams.

5.1 Background

When needed, we will consider $G(k,r)$ to be embedded into $\mathbf{P}^{\binom{r}{k}-1}$ via the Plücker embedding.

For convenience, we recall here the definition of the Schubert variety Ω_D.

Definition. Let $D \subset k \times (r-k)$ be a diagram. Then Ω_D° consists of all elements of $G(k,r)$ that can be written as the row span of a $k \times r$ matrix $A = (I_k \mid B)$, where I_k is the identity matrix and B is a $k \times (r-k)$ matrix such that $B_{ij} = 0$ if $(i,j) \in D$. The *Schubert variety* Ω_D is the closure of Ω_D°. We will denote its homology class in $H_*(G(k,r))$ by $[\Omega_D]$ and the Poincaré dual cohomology class in $H^*(G(k,r))$ by σ_D.

Given two homology (resp. cohomology) classes X and Y of the same dimension, we will write $X \geq Y$ if $X - Y$ can be written as a nonnegative linear combination of $[\Omega_\lambda]$ (resp. σ_λ).

Remember that $D^\vee$ is the complement of D within $k \times (r-k)$. It will often be more convenient to consider $\Omega_{D^\vee}$ instead of Ω_D. (For instance, $\dim \Omega_{D^\vee}$ is the number of boxes in D, and $\deg \Omega_{\lambda^\vee} = f^\lambda$.) For this reason, let us write $\sigma_{D^\vee} = \sum_\lambda d_\lambda^D \sigma_{\lambda^\vee}$. Then our main conjecture states (using Proposition 1.2.1) that $c_\lambda^D = d_\lambda^D$.

Consider the isomorphism $G(k,r) \to G(r-k,r)$ given by orthogonal complement. Then for $A = (I_k \mid B) \in \Omega_D^\circ$, $A^\perp = (-B^T \mid I_{r-k})$. It follows that the corresponding isomorphism between $H^*(G(k,r))$ and $H^*(G(r-k,r))$ sends $\sigma_D \mapsto \sigma_{D^T}$, and hence $d_\lambda^D = d_{\lambda^T}^{D^T}$.

Note that if we take $\mathbf{C}^r \subset \mathbf{C}^{r+1}$ as the hyperplane where the last coordinate vanishes, this gives an embedding $i \colon G(k,r) \hookrightarrow G(k,r+1)$. Then the image of $\Omega_{D^\vee} \subset G(k,r)$ is $\Omega_{D^\vee} \subset G(k,r+1)$. It follows that d_λ^D does not depend on r, and by transposing, we see that it does not depend on k either.

In fact, $d_{\lambda^\vee}^{D^\vee}$ also does not depend on the size of the ambient rectangle.

Proposition 5.1.1. *The coefficients $d_{\lambda^\vee}^{D^\vee}$ do not depend on the choice of k and r as long as $D \subset k \times (r-k)$.*

Proof. For clarity, denote D by $D(r) \subset k \times (r-k)$ or $D(r+1) \subset k \times (r-k+1)$ depending on the size of the ambient rectangle. As above, it suffices to show that $d_{\lambda^\vee}^{D(r)^\vee} = d_{\lambda^\vee}^{D(r+1)^\vee}$.

Note that $i(\Omega_{D(r)}) = \Omega_{D(r+1)} \cap i(G(k,r))$, and the intersection is generically transverse. It follows that $i^*\sigma_{D(r+1)} = \sigma_{D(r)}$, and therefore $d_{\lambda^\vee}^{D(r)^\vee} = d_{\lambda^\vee}^{D(r+1)^\vee}$ whenever $\lambda \subset k \times (r-k)$. To complete the proof, we need that $d_\lambda^{D(r+1)} = 0$ whenever $\lambda \subset k \times (r-k+1)$ but $\lambda \not\subset k \times (r-k)$. Therefore, let us assume that λ contains a part of size $r-k+1$ and that $|\lambda| + |D| = k(r-k+1)$.

Let $\mu \subset (k-1) \times (r-k+1)$ be λ with the part of size $r-k+1$ removed. Then we can embed $\Omega_{\mu^\vee} \subset G(k-1,r) \hookrightarrow G(k-1,r+1)$ by embedding $\mathbf{C}^r \hookrightarrow \mathbf{C}^{r+1}$ as the hyperplane where the kth coordinate vanishes. Then $\Omega_{\lambda^\vee}$ consists of all k-subspaces that contain an element of $\Omega_{\mu^\vee}$. In particular, for any $V \in \Omega_{\lambda^\vee}$, there exists a $(k-1)$-subspace $V' \subset V$ such that any k-subspace containing V' lies in $\Omega_{\lambda^\vee}$. Applying the dual argument to $D(r+1)$, we find that for any $V \in \Omega_{D(r+1)}$, there exists a $(k+1)$-subspace $V'' \supset V$ such that any k-subspace contained in V'' lies in

$\Omega_{D(r+1)}$.

Now let $Y = \Omega_{D(r+1)} \cap g \cdot \Omega_{\lambda^\vee}$ for $g \in GL(r+1)$. By Bertini-Kleiman [10], this intersection will have dimension 0 for generic g. Suppose Y is nonempty, and choose $V \in Y$. Then by above, there exist $V' \subset V \subset V''$ such that any subspace containing V' and contained in V'' lies in the intersection also. But then Y has dimension at least 1, which is a contradiction. Thus Y is empty, so $d_{\lambda^\vee}^{D(r+1)^\vee} = \sigma_{D(r+1)} \smile \sigma_{\lambda^\vee} = 0$. $\square$

Suppose G_1 and G_2 are bipartite graphs with disjoint vertex sets, and let $G = G_1 + G_2$. In terms of diagrams, G_1 and G_2 have disjoint row and column sets, and G is their union. Then just as $s_G = s_{G_1} s_{G_2}$, we can also say something about $\sigma_{G^\vee}$.

Proposition 5.1.2. *Let G_1 and G_2 be (2-colored) bipartite graphs, and let $G = G_1 + G_2$ be their disjoint union. Then*

$$d_\lambda^G = \sum_\mu \sum_\nu c_{\mu\nu}^\lambda d_\mu^{G_1} d_\nu^{G_2}.$$

In other words,

$$\sum_\lambda d_\lambda^G s_\lambda = \left(\sum_\mu d_\mu^{G_1} s_\mu \right) \left(\sum_\nu d_\nu^{G_2} s_\nu \right).$$

Proof. Write $\Omega_{G_1^\vee} \subset G(k_1, r_1)$ and $\Omega_{G_2^\vee} \subset G(k_2, r_2)$, and let $k = k_1 + k_2$, $r = r_1 + r_2$. We can embed $\mathbf{C}^{r_1}$ and $\mathbf{C}^{r_2}$ as complementary subspaces of $\mathbf{C}^r$. Consider the map that sends $(V_1, V_2) \in G(k_1, r_1) \times G(k_2, r_2)$ to their span $V = V_1 + V_2 \in G(k, r)$. (The nonzero Plücker coordinates of V are just products of Plücker coordinates of V_1 and V_2). We claim that the induced map on homology

$$H_{n_1}(G(k_1, r_1)) \otimes H_{n_2}(G(k_2, r_2)) \to H_{n_1+n_2}(G(k, r))$$

sends $[\Omega_{\mu^\vee}] \otimes [\Omega_{\nu^\vee}] \mapsto \sum_\lambda c_{\mu\nu}^\lambda [\Omega_{\lambda^\vee}]$. The result will then follow by linearity.

The subset of all k-subspaces of $\mathbf{C}^r$ that contain a subspace in $\Omega_{\mu^\vee} \subset G(k_1, r_1)$ is itself the Schubert variety $\Omega_{\tilde{\mu}^\vee} \subset G(k, r)$, where $\tilde{\mu}$ is the partition formed from μ by adding k_2 parts of size $r - k$. Likewise, if $\tilde{\nu}$ is the partition formed from ν by adding k_1 parts of sive $r - k$, then $\Omega_{\tilde{\nu}^\vee} \subset G(k, r)$ consists of all k-subspaces containing a subspace in $\Omega_{\nu^\vee}$. These two Schubert varieties intersect generically transversely, and we wish to show that the class of their intersection is $\sum_\lambda c^\lambda_{\mu\nu}[\Omega_{\lambda^\vee}]$. We therefore need only check that $c^{\lambda^\vee}_{\tilde{\mu}^\vee\tilde{\nu}^\vee} = c^\lambda_{\mu\nu}$ for $\lambda \subset k \times (r-k)$.

But this follows from properties of Littlewood-Richardson coefficients: recall that $c^{\lambda^\vee}_{\tilde{\mu}^\vee\tilde{\nu}^\vee} = c^{\tilde{\nu}}_{\tilde{\mu}^\vee\lambda}$, which is the coefficient of s_λ in $s_{\tilde{\nu}/\tilde{\mu}^\vee}$. But the skew diagram $\tilde{\nu}/\tilde{\mu}^\vee$ consists of the disjoint union of μ (rotated a half-turn) and ν. Thus $s_{\tilde{\nu}/\tilde{\mu}^\vee} = s_\mu s_\nu$, and the coefficient of s_λ is $c^\lambda_{\mu\nu}$, as desired. $\square$

An alternate way to define $\Omega_{D^\vee}$ is as the image of a certain rational map. Specifically, note that if B has entries b_{ij}, where $1 \leq i \leq k$, $1 \leq j \leq r-k$, and $b_{ij} = 0$ if $(i,j) \notin D$, then the Plücker embedding sends $A = (I_k \mid B)$ to the minors of B (up to sign), which are polynomials in the nonzero b_{ij}. Therefore $\Omega^\circ_{D^\vee}$ is the image of the map from affine space $\mathbf{A}^{|D|}$, which has coordinates b_{ij} for $(i,j) \in D$, given by these minors. In particular, $\Omega_{D^\vee}$ is an irreducible rational variety, and its projective coordinate ring $R(\Omega_{D^\vee})$ is isomorphic to the $\mathbf{C}$-algebra generated by the minors of B (appropriately homogenized).

5.2 Deformation

In this section, we informally describe a deformation strategy that will allow us to perform calculations with these Schubert varieties. For more on the background needed for this section, see, for instance, [7].

Recall that given a smooth algebraic variety X, we say that subvarieties Y and Z are *rationally equivalent* if there exists a flat family $F \subset X \times \mathbf{P}^1 \to X$ for which both Y and Z appear as fibers. The *Chow group* $A^k(X)$ then consists of formal linear combinations of codimension k subvarieties of X up to rational equivalence, and the *Chow ring* $A^*(X) = \bigoplus_k A^k(X)$ is endowed with a multiplication structure given by intersection. It is well known that there is a natural map $f\colon A^*(X) \to H^{2*}(X)$ (taking a subvariety to the Poincaré dual of its homology class), and that when $X = G(k,r)$, this map is an isomorphism.

Therefore, one approach to computing the class of an irreducible subvariety $Y \subset$

$G(k,r)$ is to construct a flat family in $Y \times \mathbf{P}^1$ (or $Y \times \mathbf{A}^1$) such that the general fiber is Y but one special fiber is reducible. This would then show that the class of Y is the sum of the classes of each reducible component. One could then try to iterate this procedure until the class of each component is evidently a Schubert cycle σ_λ.

As a simple example to demonstrate the method, let $V_1 = \langle e_1, e_4 \rangle$ and $V_2 = \langle e_2, e_3 \rangle$ be subspaces of $\mathbf{C}^4$. Then the set of 2-dimensional subspaces in $G(2,4)$ that are spanned by a vector in V_1 and a vector in V_2 is Ω_D, where D is the skew diagram $(2,1)/(1)$ consisting of two disjoint boxes. It is easy to check that $R(\Omega_D) = \mathbf{C}[p_{12}, p_{13}, p_{24}, p_{34}]/(p_{13}p_{24} - p_{12}p_{34})$, where p_{ij} is Plücker coordinate coresponding to the minor with columns i and j. In particular, Ω_D is a quadric surface.

Define subspaces $V_1(t) = \langle u_1(t), u_4(t) \rangle$ and $V_2(t) = \langle u_2(t), u_3(t) \rangle$, where $u_3(t) = te_3 + (1-t)e_4$, and $u_i(t) = e_i$ for $i \neq 3$, and likewise define $\Omega_D(t)$, $t \neq 0$, so that it consists of matrices of the form

$$\begin{pmatrix} a & 0 & 0 & d \\ 0 & b & ct & c(1-t). \end{pmatrix} \qquad (*)$$

For $t \neq 0$, $\Omega_D(t)$ and Ω_D are linearly equivalent. Then $\Omega_D(t)$, $t \neq 0$, is given by the ideal

$$I = (p_{13}p_{24} - p_{12}p_{34}, tp_{14} + tp_{13} - p_{13}).$$

Let $\Omega_D(0)$ denote the flat limit of $\Omega_D(t)$ as $t \to 0$. To compute this, one needs to find the saturation of I with respect to t and take the limit as $t \to 0$. (See [4].) In this case, it turns out that the ideal of the limit $\Omega_D(0)$ is given simply by setting $t = 0$ in the expression above. It follows that $\Omega_D(0)$ has projective coordinate ring $\mathbf{C}[p_{12}, p_{24}, p_{34}]/(p_{12}p_{34})$. This is the union of two subspaces, one where $p_{12} = p_{13} = p_{23} = 0$ and the other where $p_{13} = p_{23} = p_{34} = 0$. One can check that these are (linearly equivalent to) $\Omega_{(2)}$ and $\Omega_{(1,1)}$, respectively. Therefore, $\sigma_D = \sigma_{11} + \sigma_2$.

In general, the calculation is rather complicated because finding the ideal I and its saturation usually involves computing Gröbner bases. One thing that is easier to do is to apply the same strategy but instead of computing $\Omega_D(0)$ directly, merely show that it contains a certain component. This gives an inequality instead of an equality, but it will still sometimes be enough for our purposes. If needed, it is often possible to apply an ad hoc argument to show that equality holds.

For instance, simply by setting $t = 0$ in $(*)$, we get that all matrices of the form $\left(\begin{smallmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & c \end{smallmatrix}\right)$ must lie in $\Omega_D(0)$. The closure of these matrices forms a component in $\Omega_D(0)$ with class σ_{11}.

Likewise, by choosing appropriate values of a, b, c, and d in $(*)$, we see that $\Omega_D(t)$ contains

$$\begin{pmatrix} a' & 0 & 0 & -\frac{1-t}{t} \\ 0 & t \cdot b' & t & 1-t \end{pmatrix}$$

for constants a' and b' when $t \neq 0$. Adding t^{-1} times the second row to the first and taking the limit as $t \to 0$ shows that $\left(\begin{smallmatrix} a' & b' & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$ lies in $\Omega_D(0)$. This shows that $\Omega_D(0)$ contains a component with class σ_2. We can therefore conclude that $\sigma_D \geq \sigma_{11} + \sigma_2$, and a degree check shows that equality must hold.

We will use this strategy in the next section to show that certain Schubert varieties can be "split" into irreducible pieces that are also Schubert varieties.

5.3 Recurrences

In this section, we derive several recurrences, paying special attention to their similarities to the results of Chapter 2.

The first recurrence is the direct analogue to Proposition 2.3.2.

Proposition 5.3.1. *Let U be a special transversal of a diagram D with n boxes. Then*

$$\sigma_{D^\vee} \smile \sigma_1 \geq \sum_{x \in U} \sigma_{(D \setminus x)^\vee}.$$

In particular, under the Plücker embedding,

$$\deg \Omega_{D^\vee} \geq \sum_{x \in U} \deg \Omega_{(D \setminus x)^\vee}.$$

Proof. Points in $\Omega^\circ_{D^\vee}$ can be represented by matrices $A = (I_k \mid B)$ where $b_{ij} = 0$ if $(i,j) \notin D$. Consider the minor of B corresponding to the rows and columns of U. Since U is a special transversal, this is, up to sign, just $\prod_{(i,j)\in U} b_{ij}$. It is also, up to sign, a Plücker coordinate of A. Let H be the hyperplane on which this coordinate vanishes. Then $H \cap \Omega^\circ_{D^\vee}$ has u components, one on which each b_{ij} vanishes for $(i,j) \in U$. But these components are just $\Omega^\circ_{(D\setminus x)^\vee}$ for $x \in U$. Since the hyperplane class in $G(k,r)$ is σ_1, the result follows easily. $\square$

Note that we do not necessarily have equality, for $H \cap \Omega^\circ_{D^\vee}$ could (and often will) contain an entire component in $\Omega_{D^\vee} \setminus \Omega^\circ_{D^\vee}$.

Next, we apply the deformation strategy described in the previous section to formulate a version of Lemma 2.2.4 for Schubert varieties.

In order to ease the notation, fix k and r. Given a diagram $D \subset k \times (r-k)$, write $\tilde{D} \subset k \times r$ for the diagram with boxes (i,i), $1 \le i \le k$, and whose last $r-k$ columns are a copy of D. In other words, $\tilde{D}$ marks the nonzero entries in $A = (I_k \mid B)$ for a generic element $A \in \Omega_{D^\vee}$.

For any diagram $E \subset k \times r$, let us write X°_E for the elements of $G(k,r)$ given by those matrices A with $A_{ij} = 0$ for $(i,j) \notin E$. (We may sometimes abuse notation and say that A lies in X°_E if this condition holds.) Note that X_E is irreducible, for it is the image of a rational map from $\mathbf{A}^{|E|}$.

Let X_E be the closure of X°_E. Then it is easy to see that $X_{\tilde{D}} = \Omega_{D^\vee}$. However, in general the X_E are not as well behaved as the $\Omega_{D^\vee}$. For instance, the dimension of E is not given just by counting boxes of E, and different diagrams E can yield the same X_E.

Proposition 5.3.2. *Suppose $E \subset k \times r$ is a diagram that contains two rows i_1 and i_2 such that there is a unique box $(i_1, j) \in E$ for which $(i_2, j) \notin E$. Let $F = E \cup \{(i_2, j)\}$. Then $X_E = X_F$.*

Proof. Clearly $X_E \subset X_F$. For a general matrix A in X°_F, $A_{i_1 j}$ is nonzero. But then subtracting $A_{i_2 j}/A_{i_1 j}$ times row i_1 from row i_2 yields a matrix in X°_E that represents the same point of $G(k,r)$. The result follows easily. $\square$

Given two standard basis vectors e_{j_1} and e_{j_2}, define $X_E(t)$, $t \neq 0$, to be the subvariety obtained by applying the linear transformation φ_t to X_E, where $\varphi_t(e_{j_1}) = te_{j_1} + (1-t)e_{j_2}$ and $\varphi_t(e_j) = e_j$ for all other j. Then write $\lim_{j_1 \to j_2}(X_E)$ for the flat limit of the $X_E(t)$ as $t \to 0$.

In general, it is difficult to describe $\lim_{j_1 \to j_2}(X_E)$, and it will usually not have components of the form X_F for diagrams F. However, in certain cases, we can say something interesting.

Proposition 5.3.3. *Let $E \subset k \times r$, and let F be the diagram obtained from E by collapsing columns j_1 and j_2 (as described in Chapter 2). Then $X_F \subset \lim_{j_1 \to j_2}(X_E)$. In particular, if $\dim E = \dim F$, then $[X_E] \geq [X_F]$.*

Proof. Assume without loss of generality that F is obtained from E by moving the boxes (i, j_1) to (i, j_2) for $1 \leq i \leq p$.

Let $A = (a_{ij})$ be a matrix in X_F°. Then let $A(t)$, $t \neq 0$, be the matrix $\varphi_t^{-1}(A)$ except that for $1 \leq i \leq p$, $A(t)_{ij_1} = a_{ij_2}$ and $A(t)_{ij_2} = 0$. It is easy to check that all entries of $A(t)$ outside E are 0, so $\varphi_t(A(t))$ defines an element of $X_E(t)$. But written in the standard basis, $\varphi_t(A(t))$ is identical to A except that for $1 \leq i \leq p$, $\varphi_t(A(t))_{ij_1} = ta_{ij_2}$ and $\varphi_t(A(t))_{ij_2} = (1-t)a_{ij_2}$, respectively. Letting $t \to 0$, we find that A is the limit of $\varphi_t(A(t))$, so A lies in $\lim_{j_1 \to j_2}(X_E)$. $\square$

Recall that in Lemma 2.2.4, we constructed for a diagram D two diagrams D^A and D^B as follows. Choose $(i_1, j_1), (i_2, j_2) \in D$ such that $(i_1, j_2), (i_2, j_1) \notin D$. Let D^A be the diagram obtained by collapsing rows i_1 and i_2. Let D_B be the diagram obtained from D by moving any box (a_i, j_1) to $(a_i, f(a_i))$ such that for $p < q$, $(a_q, f(a_p)) \in D^B$, with $f(i_1) = j_2$.

If we let $E = \tilde{D}$, then since D^B is obtained by repeatedly collapsing pairs of columns, we find that Proposition 5.3.3 implies $\sigma_{D^\vee} \geq \sigma_{(D^B)^\vee}$. Likewise, by transposing, it also shows that $\sigma_{D^\vee} \geq \sigma_{(D^A)^\vee}$. While we will not show that $\sigma_{D^\vee} \geq \sigma_{(D^A)^\vee} + \sigma_{(D^B)^\vee}$, we will show a special case of this that we will need in the next section and the next chapter.

Proposition 5.3.4. *Let $D \subset k \times (r-k)$, and let D^A and D^B be as in Lemma 2.2.4. Suppose further that any row of D containing boxes in both columns j_1 and j_2 is a superset of both row i_1 and row i_2 (as subsets of $[r-k]$). Then $\sigma_{D^\vee} \geq \sigma_{(D^A)^\vee} + \sigma_{(D^B)^\vee}$.*

Proof. Without loss of generality, let us assume that $a_i = i$, so that column j_1 contains boxes in rows $1, \ldots, s$. By definition, column j_1 of D corresponds to column $k + j_1$ of $\tilde{D}$.

Let S be the set of all i such that $(i, k+j_1), (i, k+j_2) \in \tilde{D}$. Let E be the diagram obtained from $\tilde{D}$ by adding all boxes (i, i_1) and (i, i_2) for $i \in S$. By repeated application of Proposition 5.3.2, $X_E = X_{\tilde{D}} = \Omega_{D^\vee}$.

Consider $\lim_{i_1 \to i_2} X_E$. We claim that it contains as two components X_{E_1} and X_{E_2}, where E_1 is obtained by collapsing rows i_1 and i_2 and then moving (i_2, i_2) to (i_1, i_2), and E_2 is obtained from E by collapsing columns i_1 and i_2 (that is, by moving (i_1, i_1) to (i_1, i_2)). Note that we already have $X_{E_2} \subset \lim_{i_1 \to i_2} X_E$ by Proposition 5.3.3.

Consider a generic matrix $A = (a_{ij})$ in $X^\circ_{E_1}$. Let us assume that $a_{i_1 i_2} = a_{i_2 i_1} = 1$ by scaling rows i_1 and i_2 appropriately. In general, rows i_1 and i_2 of A will look something like the following (we ignore columns where both entries are 0):

$$\begin{pmatrix} 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & a_{i_1 w} & \cdots \\ 1 & 0 & \cdots & a_{i_2 u} & \cdots & a_{i_2 v} & \cdots & a_{i_2 w} & \cdots \end{pmatrix}$$

Here, columns u, v, and w are such that E contains (i_1, u) but not (i_2, u), (i_2, v) but not (i_1, v), and both (i_1, w) and (i_2, w). Then let $A'(t)$, $t \neq 0$, be the matrix that is identical to A except these two rows are replaced by:

$$\begin{pmatrix} t & 1-t & \cdots & t \cdot a_{i_2 u} & \cdots & 0 & \cdots & a_{i_1 w} & \cdots \\ 0 & -\frac{1-t}{t} & \cdots & 0 & \cdots & a_{i_2 v} & \cdots & a_{i_2 w} - \frac{1}{t} a_{i_1 w} & \cdots \end{pmatrix}$$

Then $\varphi_t^{-1}(A'(t))$ has all of its entries outside E vanishing, so $A'(t)$ represents a point in $X_E(t)$. By adding $\frac{1}{t}$ times row i_1 to row i_2 and taking the limit as $t \to 0$, we get A. It follows that $X_{E_1} \subset \lim_{i_1 \to i_2} X_E$.

For a general A in $X^\circ_{E_2}$, note that the columns $S \cup \{i_1\}$ form a matrix of rank $|S|$, while for a general A in $X_{E_1}^\circ$, these columns form a matrix of rank $|S|+1$. Therefore $X_{E_1} \neq X_{E_2}$.

Finally, we claim that $[X_{E_1}] = [\Omega_{D^A}]$ and $[X_{E_2}] \geq [\Omega_{D^B}]$. It will follow that X_{E_1} and X_{E_2} have the same dimension as X_E and hence are two (irreducible) components of $\lim_{i_1 \to i_2} X_E$. Then we will have $\Omega_{D^\vee} = [X_E] \geq [X_{E_1}] + [X_{E_2}] \geq [\Omega_{D^A}] + [\Omega_{D^B}]$, completing the proof.

To see that $[X_{E_1}] = [\Omega_{D^A}]$, simply remove all the boxes (i, i_1) and (i, i_2) for $i \in S$ by Proposition 5.3.2 and switch columns i_1 and i_2. To see that $[X_{E_2}] \geq [\Omega_{D^B}]$, collapse columns i_2 and $k+j_2$ (which moves (i_1, i_2) to (i_1, j_2)), then collapse in sequence column $k+j_1$ with columns $k+f(i)$ for $1 \leq i < i_1$. Then switch columns i_1 and $k+j_1$ and remove all boxes (i, i_1) and (i, i_2) for $i \in S$ using Proposition 5.3.2. Finally, collapse in sequence column $k+j_1$ with columns $k+f(i)$ for $i_1 < i \leq s$. It is straightforward to check that this procedure yields the diagram $\tilde{D^B}$, as desired. $\square$

This result immediately implies the following.

Proposition 5.3.5. *Let G, G_1, and G_2 be three forests as appearing in Proposition 3.2.3 (the leaf recurrence). Then $\sigma_{G^\vee} \geq \sigma_{G_1^\vee} + \sigma_{G_2^\vee}$.*

Proof. As in Proposition 4.1.2, the diagram of G splits into G_1 and G_2 via Proposition 5.3.4. (The extra condition is trivially satisfied.) $\square$

We now have the analogues of all the results about forests that we needed to prove Theorem 4.1.1 and its corollaries in Chapter 4.

5.4 Forests

In this section we will consider $\Omega_{G^\vee}$ when $G \in \mathscr{F}$ and prove our main conjecture in this case.

One reason that this case is interesting is that $\Omega_{G^\vee}$ is in fact a toric variety. Recall that $G(k,r)$ exhibits an action of the torus $T = (\mathbf{C}^*)^r / \operatorname{diag}(\mathbf{C}^*)$ which acts by multiplication on each coordinate.

Proposition 5.4.1. *Let $G \in \mathscr{F}$. Then $\Omega_{G^\vee}$ is the closure of a torus orbit in $G(k,r)$ and hence a toric variety. In fact, it is the toric variety constructed from the matching polytope M_G.*

Proof. Number the white vertices (rows) of G by $1,\dots,k$ and the black vertices (columns) by $k+1,\dots,r$. Then for each edge $x=(i,k+j)$ of G, where i is white and $k+j$ is black, consider $v_x = e_{k+j} - e_i \in \mathbf{R}^r$. Since $G \in \mathscr{F}$, the v_x are linearly independent. (For instance, they are the elements of the graphical matroid of G.) Moreover, they form a unimodular basis of the subspace V of points $(a_1,\dots,a_r)$ for which $\sum_{i\in C} a_i = 0$ for each connected component C of G.

Consider the dual basis in $V^\vee = \langle e_1^\vee,\dots,e_r^\vee\rangle / \langle \sum_{i\in C} e_i^\vee \rangle$, and suppose $v_x^\vee = \sum a_i e_i^\vee$. Then for any $t \in \mathbf{C}^*$, consider the action of the torus element $(t^{a_1},\dots,t^{a_r})$ on a point of $\Omega^\circ_{G^\vee}$. This element sends $A = (I_k \mid B)$ to $A' = (I_k \mid B')$, where B' is the same as B except that the entry corresponding to x is multiplied by t. It follows easily that if $B_{i,k+j} \neq 0$ for any edge $(i,k+j)$ in G, then $\Omega_{G^\vee}$ is the closure of the torus orbit of A in $G(k,r)$.

By [9], $\Omega_{G^\vee}$ is a toric variety whose image under the moment map is the matroid polytope P corresponding to $A = (I_k \mid B)$. By definition, P is the convex hull of $e_S = \sum_{i\in S} e_i$, where S ranges over all k-subsets of $[r]$ such that the corresponding minor A_S is nonzero. Note that since $G \in \mathscr{F}$, by Proposition 4.1.3, A_S has the form $\pm \prod_{(i,k+j)\in M} b_{ij}$, where M is a matching of G, and all matchings M correspond to a unique nonzero minor A_S. But $e_S = (e_1 + \cdots + e_r) + \sum_{x\in M} v_x$. Since $\sum_{x\in M} v_x$ is just χ_M in the basis $\{v_x\}$, we have that P is exactly the matching polytope M_G (after a translation and unimodular change of basis). $\square$

We can now easily prove Conjecture 2 in the special case of a forest.

Proposition 5.4.2. *Let $G \in \mathscr{F}$. Then* $\dim S^G = V(G) = \deg \Omega_{G^\vee}$.

Proof. The first equality is Theorem 4.1.1. For the second, by Proposition 5.4.1 and [5], the degree of $\Omega_{G^\vee}$ is the normalized volume of its image P under the moment map, which is $V(G)$. $\square$

One can also deduce the result by noting that all the Plücker coordinates are monomials in the entries of B and either computing the degree of the Hilbert polynomial or using Bernstein's Theorem on solutions of generic polynomial systems. In all of these proofs, the appearance of the matching polytope is quite natural, unlike the proof of Theorem 4.1.1.

We can also now prove a special case of our main conjecture.

Theorem 5.4.3. *Let $G \in \mathscr{F}$. Then $d_\lambda^G = c_\lambda^G$. In other words,*

$$\sigma_{G^\vee} = \sum_\lambda c_\lambda^G \sigma_{\lambda^\vee}.$$

Proof. Define the symmetric function $s'_G = \sum_\lambda d_\lambda^G s_\lambda$. We wish to show that $s'_G = s_G$, so it suffices to show that s'_G satisfies the conditions of Proposition 4.2.1. The first condition is clear. The second is Proposition 5.1.2. For the third, Proposition 5.3.5 gives that if G, G_1, and G_2 are related as in the leaf recurrence, then $\sigma_{G^\vee} \geq \sigma_{G_1^\vee} + \sigma_{G_2^\vee}$. By Proposition 5.4.2, the degree of the two sides of this equation are $V(G)$ and $V(G_1) + V(G_2)$, which are equal by Proposition 3.2.3. Therefore we must have equality, proving the third condition. □

One can also prove this result by mimicking the proof of Theorem 4.1.1 (the key ingredients being Propositions 5.3.1 and 5.3.5), allowing one to deduce Proposition 5.4.2 from Theorem 5.4.3 rather than vice versa.

5.5 Other shapes

We mention here two other classes of shapes for which we can easily prove Conjecture 2 using essentially just Proposition 5.3.1, namely row convex shapes and Rothe diagrams of permutations.

We say a diagram D is *row convex* if the boxes in any row of D are contiguous. Write (i, l_i) and (i, r_i) for the leftmost and rightmost boxes in row i. By rearranging rows, we may assume that for $i < j$, either $l_i < l_j$, or else $l_i = l_j$ and $r_i \leq r_j$. We will call this process "standardization."

Proposition 5.5.1. *Let D be a row convex diagram with n boxes, and let U be the set of all boxes that have no box below or to the left of them. Then*

$$S^D|_{\Sigma_{n-1}} \cong \bigoplus_{x \in U} S^{D \setminus \{x\}}.$$

Proof. Note that any row convex diagram is *northwest*: that is, if $(i_1, j_2), (i_2, j_1) \in D$ with $i_1 < i_2$ and $j_1 < j_2$, then $(i_1, j_1) \in D$. This result then follows immediately from the branching rule for northwest shapes given by Reiner and Shimozono [21]. □

We say that the boxes in U are *corner boxes* of D.

Proposition 5.5.2. *Let D be a row convex diagram. Then* $\dim S^D = \deg \Omega_{D^\vee}$.

Proof. Suppose $D \subset k \times (r-k)$. We first show that $\deg \Omega_{D^\vee} \geq \dim S^D$ by induction on n, the number of boxes of D. The case $n = 0$ is trivial. For $n > 0$, let U be the set of corner boxes of D. Then it is easy to see that U is a special transversal, so by Proposition 5.3.1, $\deg \Omega_{D^\vee} \geq \sum_{x \in U} \deg \Omega_{(D \setminus \{x\})^\vee}$. But each $D \setminus \{x\}$ is still row convex, so by the induction hypothesis and Proposition 5.5.1, the right hand side is at least $\sum_{x \in U} \dim S^{(D \setminus \{x\})} = \dim S^D$.

We now show that $\deg \Omega_{D^\vee} = \dim S^D$ by induction on $k(r-k) - n$. If D is a rectangle, the result is well known. Otherwise, we claim that there exists a row convex diagram D' with corner boxes U' such that for some $y \in U'$, $D' \setminus \{y\}$ is equivalent to D. Consider the row i of D for which r_i is maximum, and take the largest such i. If $l_i \neq 1$, then let D' be the row convex diagram obtained from D by adding $(i, l_i - 1)$ and standardizing. It is easy to check that this has the desired property.

Suppose instead $l_i = 1$. Choose row i' such that $r_{i'} < r_i$ is maximum. (Since D is not a rectangle, i' exists.) Then we can permute the columns of D by moving columns $r_{i'} + 1, r_{i'} + 2, \ldots, r_i$ to the far left. After standardizing, we can find D' as in the previous case, proving the claim.

By induction, Proposition 5.5.1, the first paragraph above, and Proposition 5.3.1,

$$\deg \Omega_{(D')^\vee} = \dim S^{D'} = \sum_{x \in U'} \dim S^{D' \setminus \{x\}} \leq \sum_{x \in U'} \deg \Omega_{(D' \setminus \{x\})^\vee} \leq \deg \Omega_{(D')^\vee}.$$

Then equality must hold everywhere, and in particular, since $D = D' \backslash \{y\}$ for $y \in U'$, we must have $\dim S^D = \deg \Omega_{D^\vee}$. □

Of course, all skew Young diagrams are row convex, and in this case we can prove Conjecture 1.

Proposition 5.5.3. *Let D be the skew Young diagram λ/μ. Then $\sigma_{D^\vee} = \sum_\nu c^\lambda_{\mu\nu}\sigma_{\nu^\vee}$. In particular, $\deg \Omega_{D^\vee} = f^{\lambda/\mu}$, the number of standard Young tableaux of shape λ/μ.*

Proof. Note that in the open subset of $G(k,r)$ where the first Plücker coordinate is nonzero, $\Omega_{D^\vee}$ is the intersection of $\Omega_{\lambda^\vee}$ and Ω_μ. It follows that

$$\sigma_{D^\vee} \leq \sigma_{\lambda^\vee} \smile \sigma_\mu = \sum_\nu c^{\nu^\vee}_{\lambda^\vee\mu}\sigma_{\nu^\vee} = \sum_\nu c^\lambda_{\mu\nu}\sigma_{\nu^\vee}$$

by symmetries of Littlewood-Richardson coefficients. By Proposition 5.5.2, the degree of both sides is $f^{\lambda/\mu}$, so we must have equality. □

The second class of shapes for which we can similarly prove Conjecture 2 arise from permutations.

Let w be a permutation of $[k]$. Define the diagram $D(w)$ to consist of boxes (i,j) with $1 \leq i < j \leq k$ and $w(j) < w(i)$. The number of boxes in $D(w)$ is the number of inversions $\ell(w)$. It is known that $\dim S^{D(w)}$ is the number of *reduced decompositions* of w, and the $D(w)$ are known to satisfy the following branching rule [14].

Proposition 5.5.4. *Let w be a permutation of $[k]$ with $\ell(w) = n$. Then*

$$S^{D(w)}|_{\Sigma_{n-1}} \cong \bigoplus_v S^{D(v)},$$

where v ranges over all permutations of $[k]$ such that $v^{-1}w$ is a simple transposition and $\ell(v) = n - 1$.

In other words, let U be the set of boxes of $D(w)$ of the form $(j, j+1)$ with $w(j+1) < w(j)$. It is easy to see that U is a special transversal. Then the $D(v)$ in

Proposition 5.5.4 are equivalent (after switching rows j and $j+1$ and columns j and $j+1$) to the diagrams $D(w)\backslash\{x\}$ for $x \in U$.

We can then use the same argument as in Proposition 5.5.2 to show the following.

Proposition 5.5.5. *Let w be a permutation of $[k]$. Then* $\dim S^{D(w)} = \deg \Omega_{D(w)^\vee}$.

Proof. The proof is essentially the same as that of Proposition 5.5.2. By induction on $\ell(w)$ and Propositions 5.5.4 and 5.3.1, we find that $\dim S^{D(w)} \leq \deg \Omega_{D(w)^\vee}$.

To show that equality holds, we induct on $\binom{k}{2} - \ell(w)$. Note that when w is the long permutation $w_0 = k(k-1)\cdots 1$, $D(w_0)$ is the partition $(k-1, k-2, \ldots, 1)$, for which we already know equality holds. Otherwise, we can find some w' such that $w^{-1}w'$ is a simple transposition and $\ell(w') = \ell(w) + 1$. The result then follows as in the last paragraph of the proof of Proposition 5.5.2. $\square$

Having demonstrated a number of curious similarities between the structure of Specht modules and Schubert varieties, we will now provide an application in the form of a Littlewood-Richardson rule in the next chapter.

Chapter 6

A Littlewood-Richardson rule

In this chapter, we will present a Littlewood-Richardson rule based on an algorithm for deforming a skew diagram into a straight shape. One key feature of this Littlewood-Richardson rule is that each intermediate diagram can be interpreted as either a Specht module or a Schubert variety (for a diagram that is usually not a skew Young diagram). This rule is also a combinatorial simplification of the geometric rule introduced by Coskun [2], which he later generalizes to computing intersections of two-step flag varieties.

The results of this section have been previously published by the author in [15].

6.1 The classical rule

We first recall the classical Littlewood-Richardson rule. (See, for instance, [22].)

The *reverse reading word* of a skew Young tableau is the sequence of numbers in the tableau read in rows from top to bottom, right to left within each row. A *ballot sequence* is a sequence of positive integers such that in any initial segment of the sequence, there are at least as many occurrences of i as there are of $i+1$. A semistandard Young tableau whose reverse reading word is a ballot sequence is called a *Littlewood-Richardson tableau.*

Proposition 6.1.1 (Classical Littlewood-Richardson rule)**.** *The coefficient $c^{\lambda}_{\mu\nu}$ is the number of Littlewood-Richardson tableaux of shape λ/μ and weight ν.*

We will show that our Littlewood-Richardson rule is in bijection with this classical rule. For this purpose, it will be useful for us to have a slightly different characterization of Littlewood-Richardson tableaux.

Proposition 6.1.2. *Let T be a semistandard skew Young tableau. For any integer m, let T_m be the collection of all boxes (i, j) that contain a number greater than $i - m$. Then T_m is a semistandard skew Young tableau. Moreover, T is a Littlewood-Richardson tableau if and only if for every m, the weight of T_m is a weakly decreasing sequence.*

Proof. Suppose $(i, j) \in T \backslash T_m$. Then we cannot have $(i - 1, j) \in T_m$, for then (i, j) would contain a number at most $i - m$ but $(i - 1, j)$ would contain a number greater than $i - m - 1$, which contradicts that the columns of T strictly increase. Likewise, we cannot have $(i, j - 1) \in T_m$ since the rows of T weakly increase. It follows that T_m has skew shape and is hence a semistandard skew Young tableau.

Let T_m have weight $(\alpha_{1,m}, \alpha_{2,m}, \dots)$. Then $\alpha_{p,m}$ is the number of occurrences of p in T above row $p + m$. Note that T is a Littlewood-Richardson tableau if and only if the number of occurrences of p above row i is at most the number of occurrences of $p - 1$ above row $i - 1$. But this condition is just that $\alpha_{p,i-p} \leq \alpha_{p-1,i-p}$, which is exactly the condition that the weight of each T_m is weakly decreasing. $\square$

6.2 The main algorithm

We now present the algorithm that will form the basis of the rule. As before, we assume that our diagrams fit inside a $k \times (r - k)$ rectangle. As notation, we denote the leftmost and rightmost boxes in row i of a diagram by (i, l_i) and (i, r_i). If row i is empty, we will write $l_i = \infty$ and $r_i = 0$. For instance, given a skew Young diagram λ/μ with at least one box in row i, we find that $l_i = \mu_i + 1$ and $r_i = \lambda_i$.

Algorithm 1 is presented in Table 6.1. An example computation is given in Figure 6-1.

Note that this algorithm is non-deterministic, in that at several points there is a choice of whether to perform Step A or Step B. Our main result is that this algorithm

Input: A skew Young diagram $D = \lambda/\mu$.
Output: A partition ν.

While there exists $i > 1$ such that $l_{i-1} > l_i$, do the following:
Take such i to be maximum, and perform either Step A or Step B according to the following rules. If $r_i \geq l_{i-1} - 1$ or row $i-1$ is empty, then you may perform Step A, while if $r_i \leq r_{i-1} - 1$, then you may perform Step B. (If both conditions hold, choose one to perform.)
Step A: For each $l_i \leq j' < l_{i-1}$, switch box (i, j') with box $(i-1, j')$ in D.
Step B: For each $i' \geq i$ such that $l_i = l_{i'}$, switch box $(i', l_{i'})$ with box $(i', r_{i'} + 1)$ in D.

Final Step: Once the l_i are all weakly increasing, shift all boxes in the diagram to the right as far as possible (giving the Young diagram of a partition justified to the northeast).

Table 6.1: Algorithm 1

is a Littlewood-Richardson rule, in that applying the algorithm in all possible ways gives the Littlewood-Richardson coefficients.

Theorem 6.2.1. *The number of ways to apply Algorithm 1 to the skew diagram of shape λ/μ and finish with a diagram of shape ν is exactly $c^{\lambda}_{\mu\nu}$.*

First we give a few observations regarding Algorithm 1:

1. All intermediate diagrams are row convex. To see this, note that only Step A can affect row convexity. If row $i-1$ is empty, it simply shifts row i up. Otherwise, the condition on when Step A can be performed ensures that the new row $i-1$ contains boxes in all columns from l_i to r_{i-1}, while row i contains boxes in all columns from l_{i-1} to r_i.

2. If one ignores empty rows, all intermediate diagrams have r_i weakly decreasing. To see this, note that only Step B can affect this condition. This step then replaces $r_{i'}$ with $r_{i'}+1$ if $i' \geq i$ and $l_i = l_{i'}$. By the condition on when Step B can be performed, $r_i + 1 \leq r_{i-1}$. Therefore the only problem can occur if we change some row $i' > i$ but not row $i' - 1$. This could only happen if $l_{i'-1} \neq l_i = l_{i'}$. But by maximality of i, $l_{i'} \geq l_{i'-1} \geq l_i = l_{i'}$, so this is impossible.

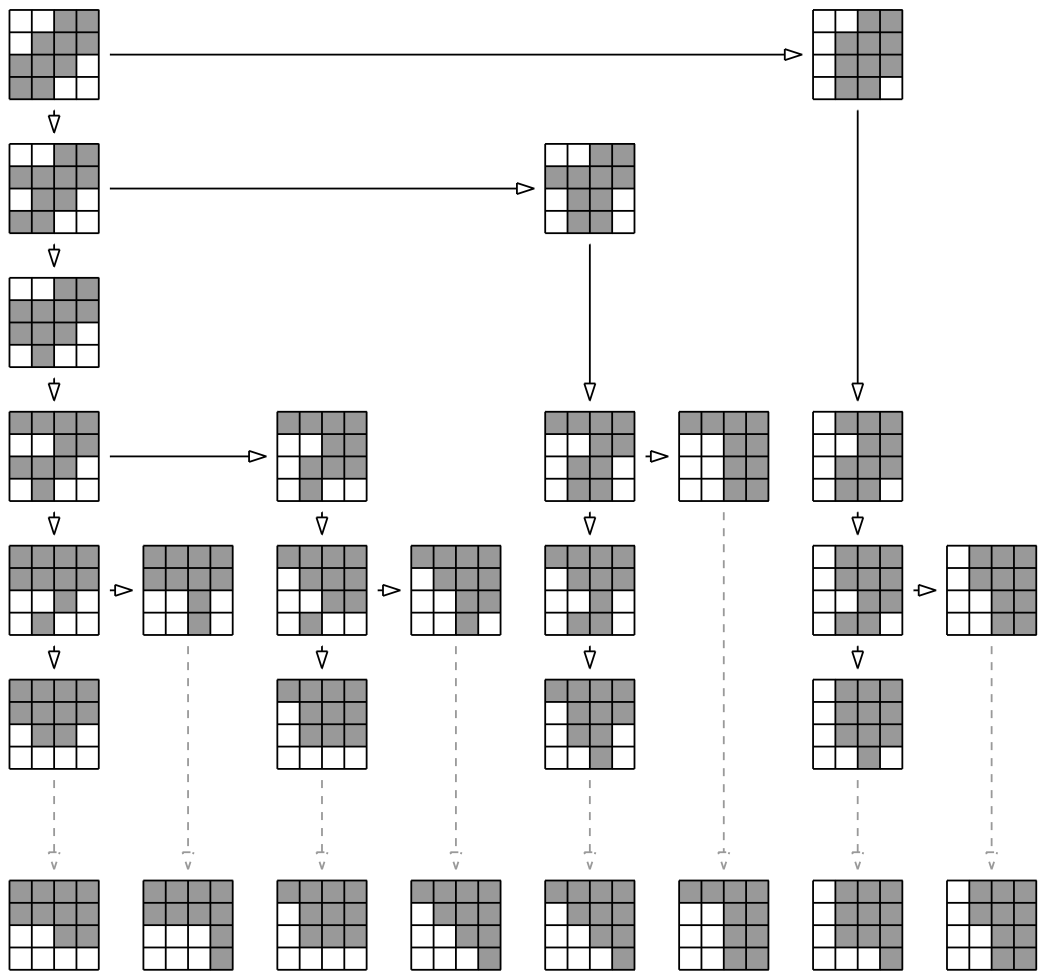


Figure 6-1: Algorithm 1 with $\lambda = (4,4,3,2)$ and $\mu = (2,1)$, showing $c^{\lambda}_{\mu\nu} = 2$ for $\nu = (4,3,2,1)$, $c^{\lambda}_{\mu\nu} = 1$ for $\nu = (4,4,2), (4,4,1,1), (4,3,3), (4,2,2,2), (3,3,3,1), (3,3,2,2)$, and $c^{\lambda}_{\mu\nu} = 0$ otherwise. Equivalently, $\sigma_{21}^2 = \sigma_{42} + \sigma_{33} + \sigma_{411} + 2\sigma_{321} + \sigma_{222} + \sigma_{3111} + \sigma_{2211}$ in $G(4,8)$. Boxes of D are shaded. Solid vertical arrows indicate applications of Step A and horizontal arrows indicate applications of Step B. The final dotted arrow in each column indicates the final step (if necessary).

3. The algorithm terminates at the Young diagram of a partition justified to the northeast. Indeed, boxes are only moved up or to the right, and no rightmost box is ever moved to the right, implying termination. Since at the end, the l_i are weakly increasing and the r_i are weakly decreasing, it follows that the result is a partition.

4. By examining the effect of Steps A and B on the l_i, one obtains the following: if one ignores empty rows, the sequence of l_i is either of the form

$$l_i \leq l_{i+1} \leq \cdots \leq l_k \leq l_{i-1} \leq l_{i-2} \leq \cdots \leq l_1$$

or else

$$l_a \leq l_{a+1} \leq \cdots \leq l_{i-2} \leq l_i \leq \cdots \leq l_k \leq l_{i-1} \leq l_{a-1} \leq l_{a-2} \leq \cdots \leq l_1.$$

In particular, if Step A is applied to rows i and $i-1$, then any two columns in which a box of D moves are identical above row $i-1$. (One can check that this still holds even if row $i-1$ was empty: if row $i-1$ became empty after applying Step A to rows $i-1$ and $i-2$, then the claim follows from the inequalities above. Otherwise it must have been empty in the original diagram, in which case any two affected columns are empty above row $i-1$ anyway.)

5. Consider any row convex diagram of boxes with no empty rows such that the r_i are weakly decreasing and the l_i satisfy one of the two inequalities of Observation 4. We call such a diagram *almost skew*. It is easy to see that any almost skew diagram occurs as an intermediate diagram when applying the algorithm to λ/μ, where λ has parts of size r_i and μ has parts of size $l_i - 1$.

6.3 The secondary algorithm

In order to prove the main result combinatorially, we will give a direct bijection to the classical Littlewood-Richardson rule. To create the necessary semistandard tableaux,

Input: A skew Young diagram $D = \lambda/\mu$ inside a $k \times (n-k)$ rectangle with boxes labeled as described above.
Output: A skew tableau of shape $\nu^\vee/\mu$ and weight $\lambda^\vee$.

While there exists a box (i, j) that is labeled but $(i-1, j)$ is unlabeled, do the following:
Take such i to be maximum, and perform either Step A or Step B according to the following rules. If $r_i \geq l_{i-1} - 1$, or if row $i-1$ or row i contains no box labeled D, then you may perform Step A, while if $0 \neq r_i \leq r_{i-1} - 1$ and $l_i < l_{i-1}$, then you may perform Step B. (If both conditions hold, choose one to perform.)
Step A: For all j', if $(i-1, j')$ is unlabeled, switch box (i, j') with box $(i-1, j')$.
Step B: For each $i' \geq i$ such that $l_i = l_{i'}$, switch box $(i', l_{i'})$ with box $(i', r_{i'} + 1)$.

Final Step: Once all unlabeled boxes are at the bottoms of their respective columns, shift all boxes labeled D to the right, keeping the rest of the row in order. The numbered boxes then form the desired tableau (in French position).

Table 6.2: Algorithm 2

we introduce a variant of Algorithm 1.

Let $D = \lambda/\mu \subset k \times (n-k)$. Consider the semistandard Young tableau with both shape and weight $\lambda^\vee$ justified to the southeast (so all boxes in row i are numbered $k+1-i$). For clarity, let us label the boxes of D with the letter D and keep the boxes of μ unlabeled.

To deal with the extra labels, we slightly modify the algorithm as given by Algorithm 2 in Table 6.2. An example computation using the modified algorithm is given in Figure 6-2.

Some additional observations regarding Algorithm 2:

6. The unlabeled boxes are always left justified within their rows, and they never switch columns. Therefore, at the end of the algorithm, the unlabeled boxes form the same shape μ that they did at the beginning of the algorithm, though now justified to the southwest. It follows that the numbered boxes fill the skew shape $\nu^\vee/\mu$.

7. Algorithm 2 acts on D in the same way as Algorithm 1, except that some steps in Algorithm 2 do not move any boxes of D and hence do not exist in Algorithm

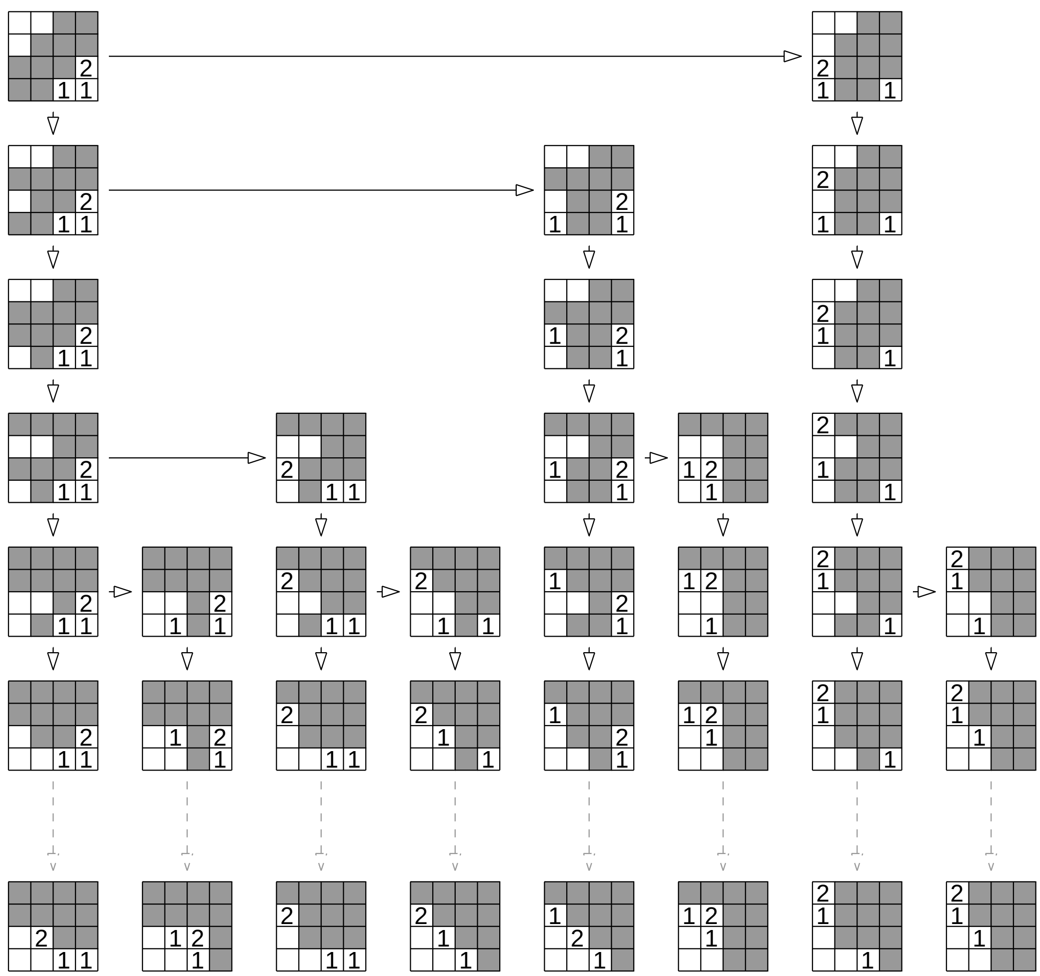


Figure 6-2: Algorithm 2 with $\lambda = (4, 4, 3, 2)$ and $\mu = (2, 1)$. For clarity, boxes labeled D are shaded. Solid vertical arrows indicate applications of Step A and horizontal arrows indicate applications of Step B. The final dotted arrow in each column indicates the final step (if necessary). Note the similarity between this figure and Figure 6-1.

1. To see that Step A acts on boxes of D in the same way in both algorithms, it suffices to show that no numbered box ever lies in the same column and above a box labeled D, and it is easy to check that this can never happen as a result of performing either Step A or Step B. To see that the extra steps do not change the number of ways to arrive at a diagram where the boxes labeled D have shape ν, it suffices to check that any time we perform Step A in Algorithm 2 without moving any boxes labeled D, we could not have performed Step B instead. But in all these cases, either row $i-1$ or row i is empty, or $l_i \geq l_{i-1}$, so only Step A is possible.

8. Note that the unlabeled boxes are only moved during Step A, and that each such step swaps the unlabeled boxes in two rows. Then notice that (ignoring Step B) the occurrences of Step A are always the same, and they occur in the same order: if μ has s nonzero parts, then Step A is always performed first for $i = s+1, s+2, \ldots$, the last of which produces an intermediate stage S_s. Then it is performed for $i = s, s+1, \ldots$, resulting in a stage S_{s-1}, then for $i = s-1, s, \ldots$, resulting in S_{s-2}, and so forth. Here, the ellipses indicate that i increases either to k or until rows i and $i+1$ contain the same number of unlabeled boxes. These intermediate stages will be important in Lemma 6.4.1 below.

6.4 The proof

The key step in the proof of the main theorem is given in the following lemma.

Lemma 6.4.1. *Let m be a positive integer. Suppose that we apply Algorithm 2 to a diagram D of shape λ/μ to arrive at a tableau T. Consider the first intermediate stage S_m at which row p contains μ_{k+m-p} unlabeled boxes for all $k+m-s \leq p \leq k$, where s is the number of nonzero parts of μ. Construct a skew tableau $T^{(m)}$ of some shape $\rho^{(m)}/\mu$ such that the numbers in row i of $T^{(m)}$ are exactly the numbers appearing in the first μ_m columns of row $k+m-i$ of S_m (in the same order that they appear).*

Then $T^{(m)}$ is semistandard, as is T. Moreover, $T^{(m)} = T\backslash T_m$, where T_m is defined as in Proposition 6.1.2.

Proof. Note first that when m is larger than s, S_m is just the initial diagram, and $T^{(m)}$ is empty. Moreover, the other S_m occur immediately after a specific Step A, as is described in Observation 8. The tableau $T^{(m)}$ is essentially constructed by considering only boxes in the first μ_m columns and the last $k-m+1$ rows, moving all numbered boxes to the left of boxes labeled D in the first μ_m columns and reindexing. Since no numbered box appears above a box labeled D, it follows that the resulting tableau has skew shape. For convenience, we let $T^{(0)} = T$.

We claim that $T^{(m-1)}$ is obtained from $T^{(m)}$ by adding boxes numbered $i-m+1$ to row i. It will then follow by an easy induction that $T^{(m)}$ is semistandard with all numbers in row i at most $i-m$. The claim that $T^{(m)} = T\backslash T_m$ will also follow immediately by the definition of T_m.

To prove the claim, note that any numbered box that does not lie in the first μ_m columns of S_m has never been moved; therefore if it lies in row i, then it is numbered $k+1-i$. Note that Step B does not change the row of any numbered box, and any numbered box that ends up in the first μ_{m-1} columns of S_{m-1} will have been moved up by exactly one row by some occurrence of Step A. Thus a box in row i of $T^{(m-1)}$, which corresponds to a box in row $k+m-i-1$ of S_{m-1}, came from row $k+m-i$ in S_m. If it came from the first μ_m columns of S_m, then it came from row i of $T^{(m)}$; otherwise, it was numbered $k+1-(k+m-i) = i-m+1$, as desired. (The claim for $m = 1$ is essentially the same.) $\square$

It follows immediately from the previous lemma that any output of Algorithm 2 is a Littlewood-Richardson tableau.

Lemma 6.4.2. *Every Littlewood-Richardson tableau of shape $\nu^\vee/\mu$ and weight $\lambda^\vee$ is uniquely obtainable from Algorithm 2.*

Proof. To show that every Littlewood-Richardson tableau T is obtainable, it suffices to show as in the previous lemma that at each intermediate stage S_m corresponding to a tableau $T^{(m)}$ as in the lemma, we can reach the desired intermediate stage S_{m-1} corresponding to a tableau $T^{(m-1)}$.

Note that for any Littlewood-Richardson tableau T, the only boxes in $T^{(m)}$ lie in the first μ_m columns. (In order for a box (i, j) to contain a number at most $i - m$, there must be at least m boxes of μ in column j.) We also note that $T^{(m)}$ must contain all but possibly $n - k - \mu_m$ instances of any number i. In other words, there can be at most $n - k - \mu_m$ instances of i appearing above row $i + m$. If $i = 1$, this is clear, because the only boxes of T above row $m+1$ lie in the last $n-k-\mu_m$ columns. But if the claim holds for i, then it immediately holds for $i + 1$ by the ballot word condition, and so the claim holds by induction.

Now consider S_m. To reach S_{m-1}, we will perform a number of instances of Step A at rows $m, m + 1, \ldots$ with some instances of Step B in between. The numbered boxes appearing in $T^{(m-1)}$ will consist of all numbered boxes appearing in the first μ_{m-1} columns of S_m along with all numbered boxes moved by these instances of Step B. As seen above, the numbers in the first μ_{m-1} columns must lie in $T^{(m-1)}$.

Algorithm 2 builds $T^{(m-1)}$ from $T^{(m)}$ by adding the columns of $T^{(m-1)} \backslash T^{(m)}$ from left to right. The instances of Step B that move boxes from the last $n - k - \mu_m$ columns of the diagram serve to insert these numbers as a column of $T^{(m-1)} \backslash T^{(m)}$; the maximum of the inserted numbers decreases by 1 with each instance of Step A that is performed. (From this, it is clear that Algorithm 2 cannot produce the same tableau in two different ways.) We need to show that when we need to insert a column of $T^{(m-1)} \backslash T^{(m)}$, this is allowed by the condition on when we can perform Step B.

The condition that we can perform Step B on row i when $l_i < l_{i-1}$ and $0 \neq r_i \leq r_{i-1} - 1$ means that we can perform the requisite instances of Step B whenever, first, the only boxes in $T^{(m-1)} \backslash T^{(m)}$ lie in the first μ_{m-1} columns, and second, the weight of $T_m = T \backslash T^{(m)}$ minus the weight of some leftmost columns of $T^{(m-1)} \backslash T^{(m)} = T_m \backslash T_{m-1}$ is weakly decreasing. We have shown above that the first condition always holds.

The second condition also always holds: note that each column of $T_m \backslash T_{m-1}$ contains consecutive numbers and that the maximum number in each column weakly decreases from left to right. Then it suffices to show the following: let σ and τ be two partitions with $\sigma_i \geq \tau_i$ for all i. Suppose that for some i', $\sigma_{i'} > \tau_{i'}$ but $\sigma_{i'+1} = \tau_{i'+1}$. Then for $i'' \leq i'$,

$$\sigma_1 \geq \cdots \geq \sigma_{i''-1} \geq \sigma_{i''} - 1 \geq \sigma_{i''+1} - 1 \geq \cdots \geq \sigma_{i'} - 1 \geq \sigma_{i'+1} \geq \ldots.$$

But this is obvious from the fact that the parts of σ are weakly decreasing, with the only subtlety arising from the fact that $\sigma_{i'} - 1 \geq \tau_{i'} \geq \tau_{i'+1} = \sigma_{i'+1}$. This shows that removing leftmost columns of $T_m \backslash T_{m-1}$ one at a time from T_m always keeps the weight weakly decreasing.

It follows that any Littlewood-Richardson tableau T is (uniquely) obtainable from Algorithm 2, proving the lemma. $\square$

With Lemma 6.4.2 proven, the proof of Theorem 6.2.1 is immediate.

Proof of Theorem 6.2.1. By Lemma 6.4.2, Algorithm 2 applied to λ/μ uniquely yields every Littlewood-Richardson tableau of shape $\nu^\vee/\mu$ and weight $\lambda^\vee$ for all ν. Since Algorithms 1 and 2 act on the boxes labeled D in identical ways, the number of ways to apply Algorithm 1 to λ/μ and arrive at a shape ν is exactly the number of these tableaux, which is $c_{\mu\lambda^\vee}^{\nu^\vee} = c_{\mu\nu}^{\lambda}$. $\square$

6.5 Specht modules and Schubert varieties

We will now show that it is possible to interpret the intermediate diagrams of Algorithm 1 (which we call *almost skew*) in terms of Specht modules and Schubert varieties. As a result, this rule makes sense not just from a combinatorial perspective, but also algebraically and geometrically.

Since we know that the coefficients $c_{\mu\nu}^{\lambda}$ appear in the structure of the Specht module $S^{\lambda/\mu}$ as well as in the cohomology class $\sigma_{(\lambda/\mu)^\vee}$ (by Proposition 5.5.3), the following result is easy.

Proposition 6.5.1. *Let D be an almost skew diagram. If only one of step A or step B can be performed on D, then the structure of the Specht module S^D and the* *cohomology class $\sigma_{D^\vee}$ are unchanged. If both can be performed, yielding diagrams D^A and D^B, then $S^D \cong S^{D^A} \oplus S^{D^B}$ and $\sigma_{D^\vee} = \sigma_{(D^A)^\vee} + \sigma_{(D^B)^\vee}$. Moreover, $c_\nu^D = d_\nu^D$, the number of ways to apply Algorithm 1, starting at D and ending at ν.*

Proof. Note that if only step A can be performed, then it only serves to swap two rows of D. Likewise, if only step B can be performed, it only permutes columns of D. Finally, if both steps A and B can be performed, then the resulting diagrams D^A and D^B are related to D as in Lemma 2.2.4 and Proposition 5.3.4, so $S^D \geq S^{D^A} \oplus S^{D^B}$ and $\sigma_{D^\vee} \geq \sigma_{(D^A)^\vee} + \sigma_{(D^A)^\vee}$. Then an easy induction shows that $S^D \geq \bigoplus_\nu S^\nu$ and $\sigma_{D^\vee} \geq \sum_\nu \sigma_{\nu^\vee}$, where the sum ranges over all applications of Algorithm 1 starting at D and ending at ν. Since equality holds when D is a skew diagram by Theorem 6.2.1, it must hold at all intermediate steps, giving the desired equality. $\square$

It follows that Algorithm 1 can be interpreted as a method of splitting the Specht module for a skew diagram as in Chapter 2, or it can be interpreted as a method of deforming a Schubert variety as in Chapter 5. Note also that we have indirectly shown that Conjecture 1 holds for all almost skew diagrams.

Chapter 7

Conclusion

In this thesis, we have drawn a curious connection between Specht modules and Schubert varieties for general diagrams, and we have exhibited a number of results suggesting that there should be a deeper correspondence that yet remains unexplained. We have also related Specht modules of forests to matching polytopes in an unexpected fashion and shown how a Littlewood-Richardson rule can be interpreted to answer the same question combinatorially, algebraically, and geometrically. Still, there remain a number of directions for further research, which we outline below.

The most obvious question to ask is whether Conjecture 1 or Conjecture 2 does in fact hold. (It seems unlikely that Conjecture 2 could hold without Conjecture 1, but it has the advantage of seeming easier to tackle.) The main difficulty is that both the structure of a general Specht module and the class of a general Schubert variety seem difficult to compute, but perhaps there is a shortcut that allows one to bridge the gap between them without explicitly calculating one or the other.

This is not the first attempt to use geometry to attack the Specht module question: Magyar [17] showed that Schur modules arise as a certain space of sections of a line bundle on an appropriately defined *configuration variety*. However, there does not seem to be any straightforward connection between these configuration varieties and Schubert varieties as we have defined them here to explain the conjectured relationship.

If Conjecture 1 were to hold, this would open a number of new directions. It could allow one to tackle the question of finding the structure of Specht modules through geometric means, including, but not limited to, the deformation procedure described above. Conversely, it could allow one to convert certain questions in Schubert calculus to a more algebraic form. In particular, it seems that in some cases it may be easier to determine when the split of a Schubert variety into two pieces is exact than when the split of the corresponding Specht module is exact, or vice versa.

Conjecture 1, if true, could also suggest a more general relationship between other geometric and algebraic structures. For instance, perhaps a more general class of subvarieties could be associated with a more general class of representations. Another possible direction is to investigate whether a similar result could hold not just for the symmetric group but also other Weyl groups. Alternatively, perhaps one could somehow relate subvarieties of higher flag varieties to some other sort of algebraic structure.

With respect to the question of Specht modules, the structure of a general Specht module is still not well understood, so it would be interesting to see whether the techniques we have used here can be at all generalized. We have also not given here a combinatorial description of the coefficients c_λ^D when D is a forest. Finally, it would also be interesting to ask whether a similar characterization to Proposition 4.2.1 exists for more general diagrams.

Along these lines, one could also ask when equality holds in Proposition 2.3.2. This would naturally lead one to ask whether there is a class of shapes containing both skew Young diagrams and forests for which a branching rule such as Corollary 4.1.6 holds. (For instance, a branching rule is known to hold for so-called *northwest shapes* [21].) One could then also ask whether a result such as Proposition 4.3.3 holds for these more general shapes as well.

Even though the proof of Theorem 4.1.1 is fairly simple, the relationship between Specht modules and matching polytopes for forests is still rather mysterious. It would be interesting to find a more natural explanation for the phenomenon described here. In particular, there may exist a larger class of diagrams for which one can naturally associate a polytope that behaves in a similar fashion. (Note, however, that the normalized volume of the matching polytope of a bipartite graph usually does not equal the dimension of its Specht module: for instance, if G is the cycle of length 4, $\dim S^G = 2$ but $V(G) = 4$.)

Another specific class of shapes that seems to be worth tackling is *toric shapes*, whose combinatorics relate to the quantum cohomology of the Grassmannian. Postnikov [19] conjectures that the resulting c^D_λ are the structure constants for the small quantum cohomology ring of the Grassmannian, and therefore that they are also in turn certain three-point Gromov-Witten invariants of the Grassmannian and certain triple intersection numbers in the two-step flag variety. Moreover, Knutson, Lam, and Speyer [11] describe certain subvarieties of the Grassmannian called *positroid varities* whose cohomology classes also give these intersection numbers, so it would be interesting to see how these relate to the generalized Schubert varieties as we have defined them here.

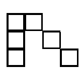
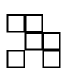
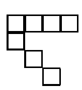
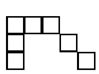
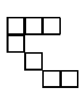
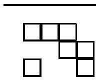
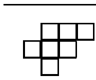
The Schubert varieties as we have defined them here have not been well studied for general diagrams, and so there remain many basic questions about them that remain unanswered. For instance, we have not shown that $d^D_\lambda = d^{D^\vee}_{\lambda^\vee}$. Another direction is to ask what can be said about the classes of these Schubert varieties in more exotic cohomology theories, such as K-theory or equivariant cohomology.

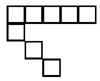
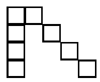
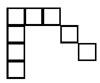
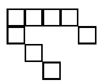
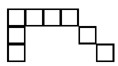
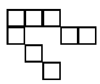
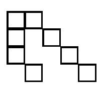
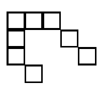
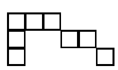
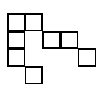
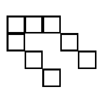
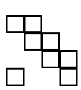
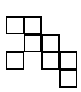
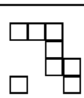
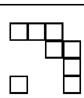
Finally, the Littlewood-Richardson rule we have given is a simplification of a special case of a more general algorithm by Coskun [2] to calculate intersections in two-step flag varieties. Therefore it makes sense to ask to what extent such a rule can be generalized to other calculations in the Grassmannian or in higher flag varieties. It may be possible to simplify and adapt other aspects of Coskun's rule to construct a similar Littlewood-Richardson rule for more general Schubert polynomials.

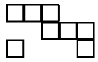
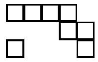
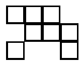
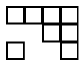
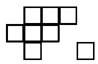
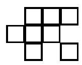
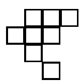
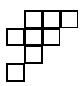
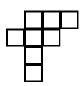
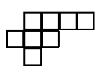
Appendix A

Computational data

The table below displays $\dim S^D$ and s_D for diagrams D with at most eight boxes. For simplicity, we display only diagrams D that are not skew and whose graphs are connected. Moreover, we only display D or its transpose D^T but not both unless of course $D = D^T$. (It is well known that $s_{D^T} = \omega(s_D)$, where $\omega\colon \Lambda \to \Lambda$ is the automorphism that sends $s_\lambda \mapsto s_{\lambda^T}$.) We have also verified computationally that $\dim S^D = \deg \Omega_{D^\vee}$ for all of these diagrams. Therefore Conjecture 2 holds for all diagrams with at most eight boxes.

D	$\dim S^D$	s_D
	47	$s_{411} + 2s_{321} + s_{222}$
	42	$s_{33} + 2s_{321} + s_{222}$
	104	$s_{43} + 2s_{421} + s_{4111}$
	127	$s_{511} + 2s_{421} + s_{331} + s_{322}$
	202	$s_{43} + 2s_{421} + s_{4111} + 2s_{331} + s_{322} + s_{3211}$
	126	$s_{43} + 2s_{421} + s_{331} + s_{322}$
	56	$s_{421} + s_{331}$

D	$\dim S^D$	s_D
	191	$s_{53} + 2s_{521} + s_{5111}$
	641	$s_{5111} + 3s_{4211} + 2s_{3311} + 3s_{3221} + s_{2222}$
	341	$s_{5111} + 2s_{4211} + s_{3311} + s_{3221}$
	491	$s_{53} + 2s_{521} + s_{5111} + s_{44} + 2s_{431} + s_{422} + s_{4211}$
	275	$s_{611} + 2s_{521} + s_{431} + s_{422}$
	575	$s_{53} + 2s_{521} + s_{5111} + 2s_{431} + s_{422} + s_{4211} + s_{332} + s_{3311}$
	1041	$s_{521} + s_{5111} + 2s_{431} + 2s_{422} + 3s_{4211} + 2s_{332} + 2s_{3311}$ $+3s_{3221} + s_{2222}$
	741	$s_{521} + s_{5111} + 2s_{431} + 2s_{422} + 2s_{4211} + 2s_{332} + s_{3311} + s_{3221}$
	387	$s_{611} + 2s_{521} + 2s_{431} + s_{422} + s_{332}$
	629	$s_{521} + s_{5111} + s_{431} + 2s_{422} + 2s_{4211} + s_{332} + s_{3311} + s_{3221}$
	987	$s_{53} + 2s_{521} + s_{5111} + s_{44} + 4s_{431} + 2s_{422} + 2s_{4211} + 2s_{332}$ $+s_{3311} + s_{3221}$
	1068	$s_{44} + 3s_{431} + 2s_{422} + 3s_{4211} + 3s_{332} + 2s_{3311} + 3s_{3221} + s_{2222}$
	726	$s_{44} + 3s_{431} + 2s_{422} + 2s_{4211} + 2s_{332} + s_{3311} + s_{3221}$
	474	$s_{431} + s_{422} + 2s_{4211} + s_{332} + s_{3311} + s_{3221}$
	432	$s_{431} + s_{422} + 2s_{4211} + s_{3311} + s_{3221}$

D	$\dim S^D$	s_D
	408	$s_{53} + 2s_{521} + s_{44} + 2s_{431} + s_{422} + s_{332}$
	282	$s_{53} + 2s_{521} + s_{431} + s_{422}$
	252	$s_{44} + 2s_{431} + s_{422} + s_{332}$
	126	$s_{431} + s_{422}$
	302	$s_{521} + 2s_{431} + s_{422} + s_{332}$
	168	$s_{431} + s_{422} + s_{332}$
	244	$s_{422} + s_{4211} + s_{332} + s_{3311}$
	314	$s_{431} + s_{422} + s_{4211} + s_{332} + s_{3311}$
	146	$s_{4211} + s_{3311}$
	134	$s_{521} + s_{431}$

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